Sign-Regular Matrices Having the Interval Property

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Outline

- Background: Systems of linear interval equations.
- A conjecture which dates back to 1982.
- Solution of the conjecture.
- Related recent work.

Notation

$\mathbb{IR}$: set of the compact, nonempty real intervals $[a] = [\underline{a}, \overline{a}]$, $\underline{a} \leq \overline{a}$,
$\mathbb{IR}^n$: set of n-vectors with components from $\mathbb{IR}$, *interval vectors*
$\mathbb{IR}^{n\times n}$: set of n-by-n matrices with cmpts. from $\mathbb{IR}$. *interval matrices*

Elements from $\mathbb{IR}^n$ and $\mathbb{IR}^{n\times n}$ may be regarded as vector intervals and matrix intervals, respectively, w.r.t. the usual entrywise partial ordering, e.g.,

$$\begin{aligned}[A] &= \Big([a_{ij}]\Big)_{i,j=1}^n = \Big([\underline{a}_{ij}, \overline{a}_{ij}]\Big)_{i,j=1}^n \\ &= [\underline{A}, \overline{A}], \quad \text{where } \underline{A} = \Big(\underline{a}_{ij}\Big)_{i,j=1}^n, \ \overline{A} = \Big(\overline{a}_{ij}\Big)_{i,j=1}^n.\end{aligned}$$

A *vertex matrix* of $[A]$ is a matrix $A = (a_{ij})_{i,j=1}^n$ with $a_{ij} \in \{\underline{a}_{ij}, \overline{a}_{ij}\}$, $i, j = 1, \ldots, n$.

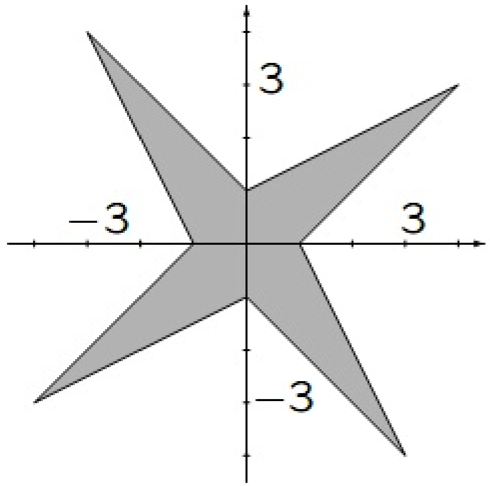
Systems of linear interval equations $[A]x = [b]$

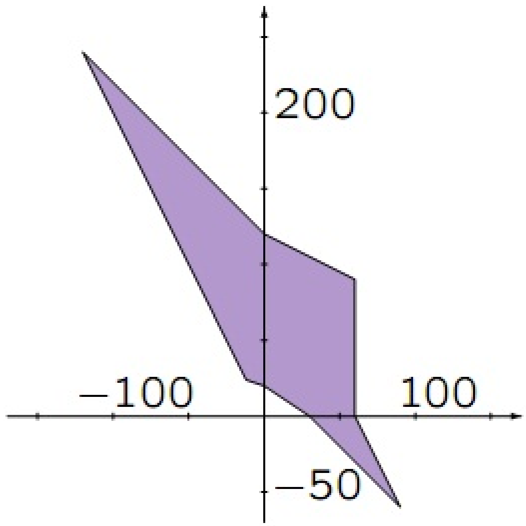
solution set $\Sigma\,([A],[b]) \;:=\; \{x \in \mathbb{R}^n \mid Ax = b,\; A \in [A],\; b \in [b]\}$

If all $A \in [A]$ are nonsingular then:

$\Sigma\,([A],[b])$ is compact, connected, and convex in each orthant.

Examples





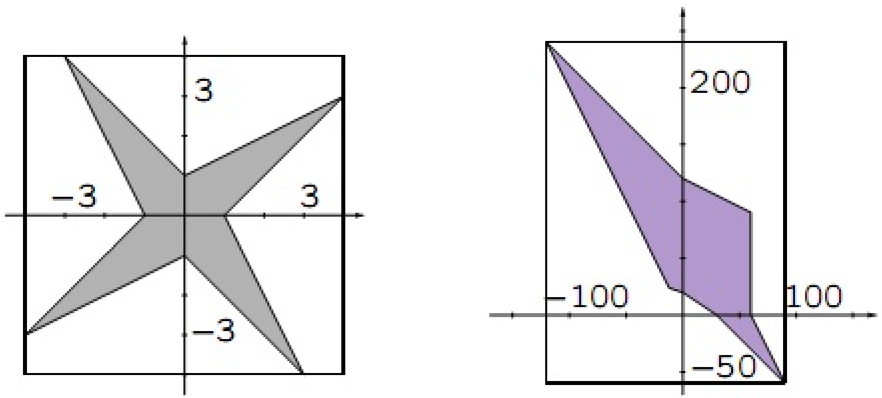
$$\begin{pmatrix} [2,4] & [-2,1] \\ [-1,2] & [2,4] \end{pmatrix} x = \begin{pmatrix} [-2,2] \\ [-2,2] \end{pmatrix} \qquad \begin{pmatrix} [2,3] & [0,1] \\ [1,2] & [2,3] \end{pmatrix} x = \begin{pmatrix} [0,120] \\ [60,240] \end{pmatrix}$$

Solution sets for Barth-Nuding and Hansen interval systems

(Interval) Hull of the solution set

$$[A]^H[b] := \square\Sigma\,([A],[b])$$

Examples (cont'd)



Interval hulls for Barth-Nuding and Hansen interval systems

Inverse nonnegative matrices

<u>Def.</u>: A is *inverse nonnegative* if $0 \leq A^{-1}$.

Proposition 1 (Kuttler, 1971)**.** *Let* $[A] = [\underline{A}, \overline{A}]$ *be a matrix interval and* $\underline{A}$ *and* $\overline{A}$ *be inverse nonnegative. Then* $[A]$ *is inverse nonnegative and* $\overline{A}^{-1} \leq \underline{A}^{-1}$.

Theorem 2 (Beeck, 1975)**.** *If* $[A] \in \mathbb{IR}^{n\times n}$ *is inverse nonnegative, then*

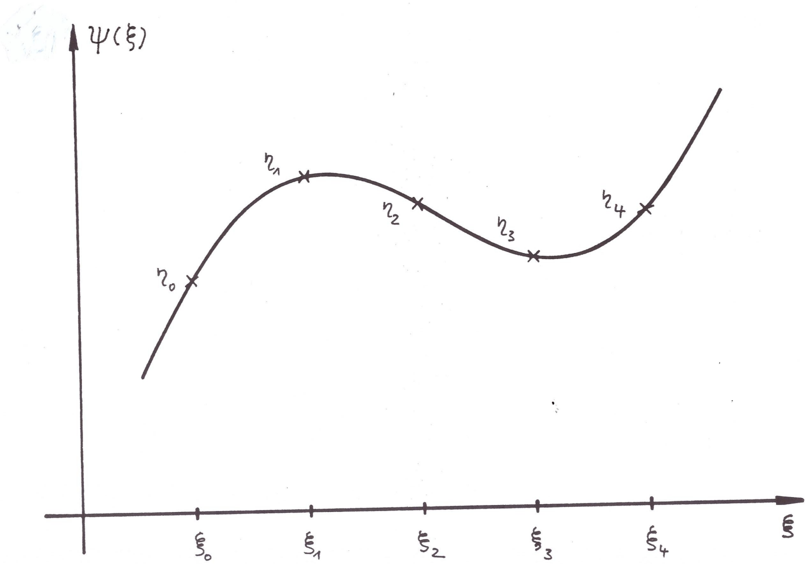
$$A^H b = \begin{cases} [\overline{A}^{-1}\underline{b},\ \underline{A}^{-1}\overline{b}] & \textit{if} \quad 0 \leq b, \\ [\underline{A}^{-1}\underline{b},\ \underline{A}^{-1}\overline{b}] & \textit{if} \quad 0 \in b, \\ [\underline{A}^{-1}\underline{b},\ \overline{A}^{-1}\overline{b}] & \textit{if} \quad b \leq 0. \end{cases}$$

In the general case, one has to solve at most $2n$ linear systems to find $\inf(A^H b)$ and similarly $\sup(A^H b)$.

(Real) Interpolation

Given: $\xi_i, \eta_i \in \mathbb{R}$, $i = 0, 1, \ldots, n$.

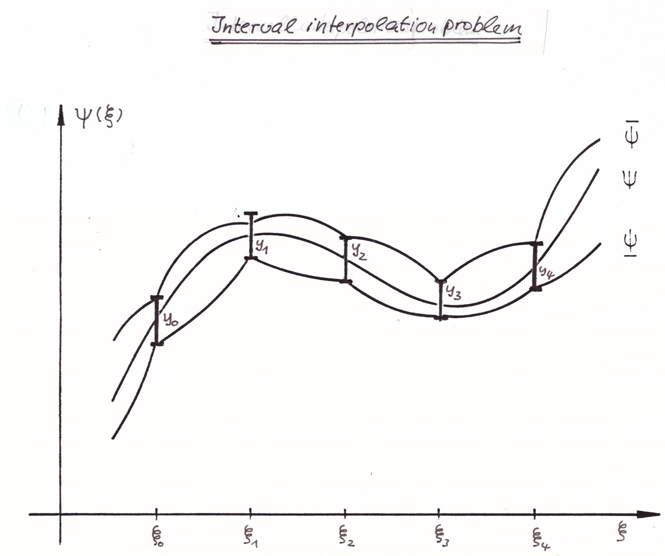
Wanted: Functions $\Psi \in \mathbb{F}$ such that $\Psi(\xi_i) = \eta_i$, $i = 0, 1, \ldots, n$.



Interval interpolation

Given: $\xi_i \in \mathbb{R}$, $[y_i] \in \mathbb{IR}$, $i = 0, 1, \ldots, n$.

Wanted: All functions $\Psi \in \mathbb{F}$ such that $\Psi(\xi_i) \in [y_i]$, $i = 0, 1, \ldots, n$.



Real interpolation problem

If $\mathbb{F}$ is the class of B-splines the collocation matrix is nonsingular and totally nonnegative.

Def. A real matrix is called *totally nonnegative* and *totally positive* if all its minors are nonnegative and positive, respectively.

S. M. Fallat and C. R. Johnson, *Totally Nonnegative Matrices*, Princeton Univ. Pr., 2011.

A. Pinkus, *Totally Positive Matrices*, Cambridge Univ. Pr., 2010.

Interval interpolation problem with B-splines

If in addition, the nodes ξ_i are uncertain, i.e., $\xi_i \in [x_i]$, $i = 0, 1, \ldots, n$, then a system of linear interval equation arises.

Question: Does the (interval) collocation matrix contain only nonsingular totally nonnegative matrices if some specified vertex matrices are nonsingular and totally nonnegative?

Totally nonnegative matrices

A suitable partial order for the totally nonnegative matrices is the *checkerboard order*. For $A, B \in \mathbb{R}^{n \times n}$ define

$$A \leq^* B := (-1)^{i+j} a_{ij} \leq (-1)^{i+j} b_{ij}, \quad i, j = 1, 2, \ldots, n.$$

This partial order is related to the usual entry-wise partial order by

$A \leq^* B \Leftrightarrow A^* \leq B^*$, where $A^* := SAS$, $S := \mathrm{diag}(1, -1, \ldots, (-1)^{n+1})$,

is the *checkerboard transformation*.

A matrix interval $[\underline{A}, \overline{A}]$ with respect to the usual entry-wise partial order can be represented as an interval $[\downarrow A, \uparrow A]^*$ with respect to the checkerboard order, where

$$(\downarrow A)_{ij} \;:=\; \begin{cases} \underline{a}_{ij} & \text{if } i+j \text{ is even,} \\ \overline{a}_{ij} & \text{if } i+j \text{ is odd,} \end{cases}$$

$$(\uparrow A)_{ij} \;:=\; \begin{cases} \overline{a}_{ij} & \text{if } i+j \text{ is even,} \\ \underline{a}_{ij} & \text{if } i+j \text{ is odd.} \end{cases}$$

If A is non-singular and totally nonnegative then $0 \leq^* A^{-1}$ and therefore, $0 \leq (A^{-1})^* = (A^*)^{-1}$. Since A^* is inverse nonnegative all results for inverse nonnegative matrices carry over to the totally nonnegative matrices by the checkerboard transformation, e.g., if A and B are non-singular and totally nonnegative then it follows that $A \leq^* B \Rightarrow B^{-1} \leq^* A^{-1}$.

Application: calculation of $[A]^H[b]$ under special sign conditions posed on $[b]$.

Conjecture (JG, 1982): If $\downarrow A$ and $\uparrow A$ are non-singular and totally nonnegative then the whole matrix interval $[\downarrow A, \uparrow A]^*$ is non-singular and totally nonnegative.

This conjecture has been settled for some subclasses of totally non-negative matrices, e.g., for the totally positive matrices, i.e., matrices having all their minors positive.

Attempts to solve the conjecture by J. Day, S. Fallat, C. R. Johnson, S. Nasserasr, E. Nuding, J. Rohn, ...

Cauchon Algorithm

We denote by $\leq$ the lexicographic order on $\mathbb{N}^2$, i. e.,

$$(g,h) \leq (i,j) :\Leftrightarrow \ (g < i) \text{ or } (g = i \text{ and } h \leq j).$$

Set $E^\circ := \{1,...,n\}^2 \setminus \{(1,1)\}$, $E := E^\circ \cup \{(n+1,2)\}$.

Let $(s,t) \in E^\circ$. Then $(s,t)^+ := min\,\{(i,j) \in E \mid (s,t) \leq (i,j),\ (s,t) \neq (i,j)\}$.

Algorithm: Let $A \in \mathbb{R}^{n,n}$. As r runs in decreasing order over the set E, we define matrices $A^{(r)} = (a_{ij}^{(r)}) \in \mathbb{R}^{n,n}$ as follows.

1. Set $A^{(n+1,2)} := A$.

2. For $r = (s,t) \in E^{\circ}$:

(a) if $a_{st}^{(r^+)} = 0$ then put $A^{(r)} := A^{(r^+)}$.

(b) if $a_{st}^{(r^+)} \neq 0$ then put

$$a_{ij}^{(r)} := \begin{cases} a_{ij}^{(r^+)} - \dfrac{a_{it}^{(r^+)} a_{sj}^{(r^+)}}{a_{st}^{(r^+)}} & \text{for } i < s \text{ and } j < t, \\ a_{ij}^{(r^+)} & \text{otherwise.} \end{cases}$$

3. Set $\tilde{A} := A^{(1,2)}$ is called the matrix obtained from A (by the Cauchon Algorithm).

If $n = 5$ and A is totally positive, then

$$\tilde{A} = \begin{bmatrix} \frac{[12345]}{[2345]} & \frac{[1234|2345]}{[234|345]} & \frac{[123|345]}{[23|45]} & \frac{[12|45]}{[2|5]} & a_{15} \\ \frac{[2345|1234]}{[345|234]} & \frac{[2345]}{[345]} & \frac{[234|345]}{[34|45]} & \frac{[23|45]}{[3|5]} & a_{25} \\ \frac{[345|123]}{[45|23]} & \frac{[345|234]}{[45|34]} & \frac{[345]}{[45]} & \frac{[34|45]}{[4|5]} & a_{35} \\ \frac{[45|12]}{[5|2]} & \frac{[45|23]}{[5|3]} & \frac{[45|34]}{[5|4]} & \frac{[45]}{[5]} & a_{45} \\ a_{51} & a_{52} & a_{53} & a_{54} & a_{55} \end{bmatrix}$$

Theorem (Goodearl, Launois, and Lenagan, '11):

1. A is totally nonnegative iff $0 \leq \tilde{A}$ and for all $i, j = 1, \ldots, n$
   $\tilde{a}_{ij} = 0 \Rightarrow \ \tilde{a}_{ik} = 0 \ \ k = 1, \ldots, j-1,$ or $\tilde{a}_{kj} = 0 \ \ k = 1, \ldots, i-1.$

$$\tilde{A} = \begin{bmatrix} & & & 0 & \\ & \text{or} & \rightarrow & \vdots & \\ & \downarrow & & 0 & \\ 0 & \ldots & 0 & \boxed{0} & \\ & & & & \\ & & & & \end{bmatrix}$$

2. If A is totally nonnegative matrix then A is nonsingular iff $0 < \text{diag}(\tilde{A})$.

Theorem*: Let A, B be nonsingular and totally nonnegative matrices and let $A \leq^* Z \leq^* B$. Then

1. $\tilde{A} \leq^* \tilde{Z} \leq^* \tilde{B}$;

2. Z is nonsingular and totally nonnegative;

3. if A, B possess the same pattern of zero minors then Z has this pattern, too.

*M. Adm and J. Garloff, Intervals of totally nonnegative matrices, *Linear Algebra and its Applications* 439 (Dec. 2013), pp. 3796-3806

The assumption of nonsingularity of certain principal minors cannot be relaxed:

$$A := \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \leq^* Z := \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix} \leq^* B := \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

totally nonnegative　　has a negative minor　　totally nonnegative

Corollary Let $A, B, Z \in \mathbb{R}^{n,n}$ with $A \leq^* Z \leq^* B$. If A, B are totally nonnegative and

$$A[2, \ldots, n] \text{ and } B[2, \ldots, n]$$

or

$$A[1, \ldots, n-1] \text{ and } B[1, \ldots, n-1]$$

are nonsingular, then Z is totally nonnegative, too.

Sign regular matrices

- Application of the Cauchon algorithm to *sign regular matrices*, i.e., matrices whose minors of the same order have the same sign or vanish.

  The interval property is established for the following subclasses:

  - nonsingular totally nonpositive matrices (all minors are non-positive),
  - tridiagonal nonsingular sign regular matrices,
  - strictly sign regular matrices (all minors of the same order have the same strict sign),
  - almost strictly sign regular matrices.

Open question: Does the interval property hold for general nonsingular sign regular matrices?

Related recent work

- Invariance of total nonnegativity under perturbation of single entries.
  (M. Adm and J. Garloff, *Operators and Matrices*, 8 (1), pp. 129-137, 2012).

- Further investigation of the application of the Cauchon algorithm to totally nonnegative matrices.
  - new determinantal criteria,
  - short proofs of properties,
  - new characterizations of subclasses.

  (M. Adm and J. Garloff, *Electronic J. Linear Algebra*, 27, pp. 588-610, 2014).