APPLICATION OF DISCRETE TRANSFORMS IN PERFORMANCE MODELING AND ANALYSIS

Anwendung zeitdiskreter Transformationen in der Nachrichtenverkehrstheorie

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Abstract *In modeling and performance analysis of modern computer and communication systems the class of discrete-time models takes an increasingly important part. For the analysis of these models methods operating in discrete-time and discrete probability environments are required. Discrete-time analysis methods can be subdivided into two main subclasses: i) analysis methods dealing directly with probability distributions in time domain and ii) analyses in transform domain. For both subclasses discrete transforms like the discrete Fourier transform (DFT) in conjunction with the fast Fourier tranform (FFT) or the Cepstrum concept are often used. The purpose of this paper is to outline the use of these discrete transform techniques, whereby the analysis of the basic queueing system of the G/G/1 type is taken as an example.*

1. Discrete-time Performance Models and Discrete Transforms

In the course of modeling modern communication systems, discrete-time model components play an increasingly important role. On the one hand, new system structures and principles often employ discrete or discretized basic time and data units. Examples are the concept of cells in asynchronous transfer mode networks (ATM) or time slots in high-speed local and metropolitan area networks (e.g., DQDB: distributed queue dual bus). On the other hand, system parameters and input values are often based on measured data, which are given in the form of histograms. They are discrete-time by nature. These facts lead to the development of discrete-time models in performance analyses of computer and communication systems, which can be observed in the recent literature.

For the analysis of this class of models, conventional methods operating in continuous time are obviously inappropriate. Due to the lack of discrete-time methods they are used in some cases in an approximate sense. In these studies equivalent continuous-time model components are employed, e.g., the discrete-time stochastic arrival and service processes are approximately described by means of random variables with well-known time-continuous types of distribution functions. A relatively small amount of studies [1-3, 10-14, 17-18, 20-24] deals with direct analysis approaches for discrete-time models. Mostly, these studies take into account the discrete-time analysis of basic queueing models like single server systems [1-3,10-13,20] or queueing networks [18]. Some other studies presented discrete-time analysis of general polling systems [22], overload control models in communication switching systems

[1] parts of the studies discussed in this paper were done while the author was with Institute of Communications Switching and Data Technique, University of Stuttgart, and IBM Zurich Research Laboratory.

[24], routing mechanisms [21] or multiplexing schemes in modern communication system architectures [23]. In these investigations the use of discrete-time transforms is an essential aspect.

In this paper we devote our attentions to the use of discrete transforms in the analysis of discrete-time stochastic models of computer and communication systems. To illustrate the application of transform techniques the analysis of the basic queueing system G/G/1 with discrete-time arrival and service processes will be discussed. Section 2 deals with iteration methods using discrete convolution algorithms. In section 3 basic functional equations and analysis algorithms in transform domains will be outlined, where the use of the Cepstrum concept in the waiting time analysis is taken as an example.

2. The G/G/1 Queueing System in Discrete-time

2.1 The G/G/1 Model and Discrete-time Lindley Equation

We consider in the following the discrete-time G/G/1 queueing system, which represents one of the important basic models used in performance studies. The term discrete-time indicates here that the time axis is slotted in equidistant time units of length Δt. The model consists of a server with a generally distributed service time and an arrival process which is a general stochastic process characterized by a generally distributed interarrival time. Arriving jobs, which find the server in a busy state, have to join an infinite capacity queue. Waiting jobs in the queue will be treated according to a service discipline, which is assumed here to be first-in, first-out (FIFO).

In the literature, a number of analysis approaches in accordance with the calculation of the waiting time distribution function of the G/G/1 queue can be found [1,8,9-14,17,19,20]. Most of these methods are related to solutions of the Lindley integral equation, which is a special form of Wiener-Hopf equations. Most of the studies consider methods operating in Laplace domain. They are based on techniques like spectral factorization, numerical poles and zeros allocation of the system function, determination of quadratic factor of polynomials [17], as well as separation of functions having convolutions in frequency domain. Ackroyd [1] presented an efficient algorithm for the calculation of the waiting time distribution of the discrete-time G/G/1 queue, where discrete transform techniques (e.g., the Cepstrum concept [4-7,26,27], phase unwrapping technique [26], etc..) and fast convolution algorithms are used. Using the same approach, a discrete-time analysis of the idle time and the interdeparture process is given in Tran-Gia [20,25].

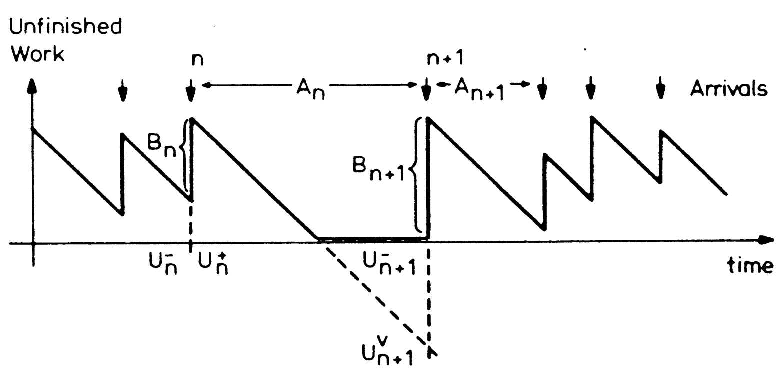

Fig. 1: Sample path of the state process of the G/G/1 model

The basic relationship for the analysis of the discrete-time G/G/1 queueing system is equivalent to the well-known Lindley integral equation for the continuous-time system [8,14,19]. It will be briefly outlined in the following. According to the discrete-time consideration, samples of the random variables are integer multiples of Δt. We assume further that the discrete distributions have finite lengths.

We observe a test job (number n) which joins the system and sees upon its arrival an amount U_n^- of unfinished work in the system (cf. Fig. 1). The service time of the test job is denoted by the random variable B_n and the interarrival time, i.e. the interval until the (n+1)-st arrival, by A_n. The corresponding distributions $a_n(k)$ and $b_n(k)$ exist for k=0,...,$N_A - 1$ and k=0,..., $N_B - 1$, respectively. The unfinished work U_n^- indicates the amount of time units the server has to work out before being able to serve the test job. Assuming the first-in, first-out (FIFO) service discipline, its waiting time W_n with the corresponding distribution (probability mass function)

$$w_n(k) = Pr\{W_n = \text{k time units of length } \Delta t\} \tag{1}$$

are identical with U_n^- and its distribution $u_n^-(k)$, respectively.

The following recursive relationship between the waiting time distributions of two successive jobs can be found [1,9,13,20]:

$$u_{n+1}^-(k) = \pi_0(u_n^-(k) * a_n(-k) * b_n(k)) \tag{2}$$

or

$$u_{n+1}^-(k) = \pi_0(u_n^-(k) * c_n(k)), \tag{3}$$

i.e.

$$w_{n+1}(k) = \pi_0(w_n(k) * c_n(k)) \tag{4}$$

with

$$c_n(k) = a_n(-k) * b_n(k). \tag{5}$$

The term $c_n(k)$ is often called the system funtion of the G/G/1 system. The symbol "*" indicates the discrete convolution operation and $\pi_0(.)$ an operator defined as follows

$$\pi_0(x(k)) = \begin{cases} x(k) & k > 0 \\ \sum_{i=-\infty}^{0} x(i) & k = 0 \end{cases} \tag{6}$$

Considering the service and arrival processes to be recurrent and the observed state process of the discrete-time G/G/1 model to be in statistical equilibrium, i.e. under stationary conditions, the index n of the test job can be suppressed. We arrive at the stationary state equation for the analysis of the waiting time distribution of the G/G/1 system:

$$w(k) = \pi_0(w(k) * c(k)). \quad (7)$$

Eqns. (4) and (7) represent discrete forms of the Lindley integral equation [14], which is well-known in the context of G/G/1 analysis.

2.2 Algorithm in Discrete-time Domain

According to eqn. (4) the waiting time distribution of the (n+1)-st job can be expressed as a function of the waiting time distribution of the n-th job and the system function. Using this fact (cf. [1,9,20]) the equilibrium waiting time distribution can be iteratively calculated, as schematically depicted in Fig. 2. The iteration procedure may start, e.g., by assuming the first customer finding an empty system.

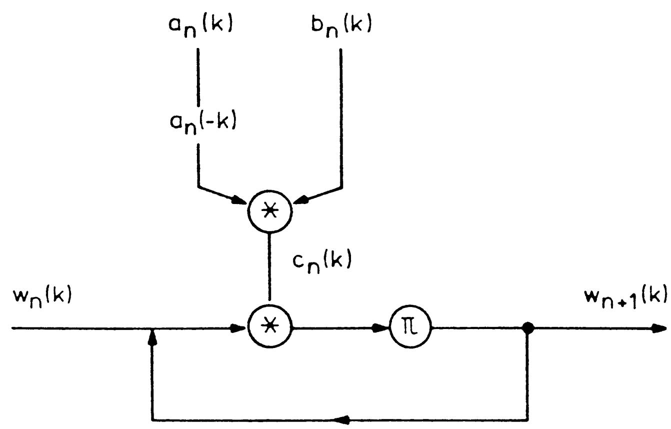

Fig. 2: Computational diagram of the algorithm in time domain

For large vector sizes of the arrival and service distributions, the discrete convolution operation can efficiently be implemented using discrete transforms and convolution algorithms, e.g., the fast Fourier transform FFT (based on the discrete Fourier transform DFT) [6,7,16]. For the analysis of a G/G/1 system under stationary conditions, the number of iteration cycles needed and in accordance with it, the computing efforts, depend strongly on the parameters of the system to be investigated. In comparison with algorithms in transform domain, e.g., the spectral factorization in Laplace- or Z-domain or the separation of maximum and minimum phase systems using the cepstrum concept (see section 3),the algorithm in probability domain (or time domain) is very robust with respect to the type of interarrival and service processes. Furthermore, the algorithm in time domain as illustrated in Fig. 2 is also applicable to G/G/1 systems with time- or state-dependent interarrival and service time distributions, e.g., systems with workload-oriented overload control [24] or G/G/1 queues with alternating input processes [21].

3. Algorithm for G/G/1 Systems in Transform Domain

3.1 Basic Relationship in Transform Domain

From eqn. (7), which was given for the *waiting time distribution* (the probability mass function), an analogous form of the waiting time *distribution function* defined by

$$W(k) = \sum_{i=-\infty}^{k} w(i) \tag{8}$$

can be obtained:

$$W^{-}(k) + W(k) = c(k) * W(k), \tag{9}$$

where $W^{-}(k)$ consists of components of the convolution $c(k) * W(k)$ lying on the negative time axis. In Z-transform domain, we obtain the following fundamental equation:

$$W_{ZT}^{-}(z). \quad \frac{1}{w_{ZT}(z)} \quad = \quad \frac{c_{ZT}(z) - 1}{1 - z^{-1}}, \tag{10}$$

where the transfer function of the G/G/1 system in Z-domain is contained:

$$S_{ZT}(z) \quad = \quad \frac{c_{ZT}(z) - 1}{1 - z^{-1}}. \tag{11}$$

It can be shown (cf.[1,25]) that for finite length distributions $a(k)$ and $b(k)$, the function $W_{ZT}^{-}(z)$ stands for the Z-transform of a maximal phase system (cf.[16]). Further, the term $\frac{1}{w_{ZT}(z)}$ corresponds to the Z-transform of a minimal phase system. This knowledge leads to solutions of eqn.(10) using pole and zero allocation schemes [13] or in conjunction with the use of the Cepstrum concept [1]. The application of the Cepstrum to the waiting time analysis of G/G/1 queueing model will be discussed below.

3.2 Algorithm using Cepstrum Concept

In the same way as in signal processing techniques, the Cepstrum concept is employed here to separate maximal and minimal phase systems [16]. Thus, the term $\frac{1}{w_{ZT}(z)}$ or consecutively, the waiting time distribution $w(k)$ in time domain, can be filtered out of the transfer function $S_{ZT}(z)$ after transforming it into the Cepstrum domain. The algorithm is illustrated in Fig. 3 containing the following major steps (cf. Ackroyd [1])

i) Calculation of the transfer function $S_{ZT}(z)$ out of the system function $c(k) = a(-k) * b(k)$. Since $c(k)$ is of finite length, $S_{ZT}(z)$ can be equivalently represented by the discrete Fourier transform (DFT) $S_{DFT}(n)$.

ii) Calculation of the complex Cepstrum

$$S_{CEP}(k) = DFT^{-1}(\ln[S_{DFT}(n)]) \tag{12}$$

iii) Separation of $S^+_{CEP}(k)$, which consists of non-negative components of $S_{CEP}(k)$. The function $S^+_{CEP}(k)$ is the Cepstrum of the unnormalized waiting time distribution $w_1(k)$

iv) Inverse transformation of $S^+_{CEP}(k)$ to get $w_1(k)$ and normalization of $w_1(k)$ to obtain finally the waiting time distribution w(k) of the G/G/1 system.

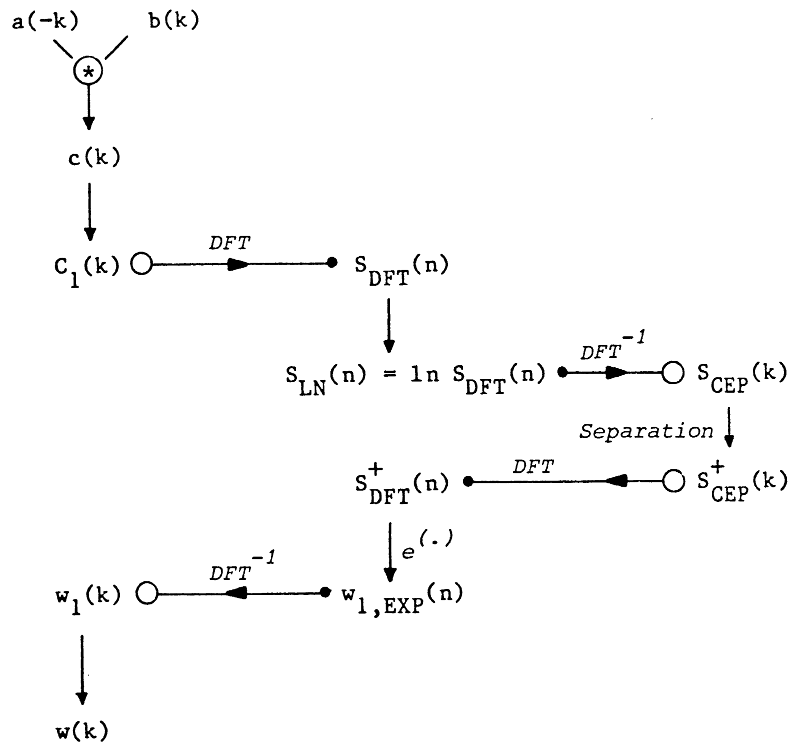

Fig. 3: The Cepstrum algorithm for G/G/1 analysis

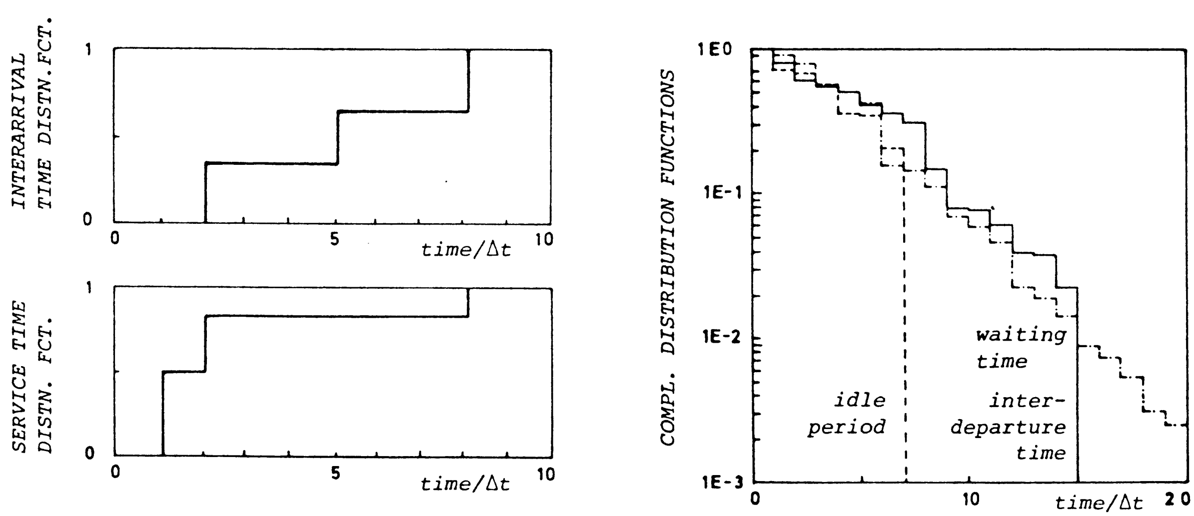

Fig. 4: Some results obtained from discrete-time analysis of a G/G/1 system

Out of the waiting time distribution, further performance measures of interest, like the idle time distribution of the service unit and the interdeparture distribution of the stochastic output process can be derived [20]. To illustrate the use of the method in the case of model parameters arising out of measurements in the form of histograms, Fig. 4 shows the distribution functions of the interarrival and the service times as well as the complementary distribution functions of the equilibrium waiting time, the idle period and the interdeparture time.

4. Concluding Remarks

As mentioned before, the aim of this paper is to provide a brief outline of the use of discrete-time transforms in the performance analysis of computer and communication systems. In such systems efficient discrete-time based analysis techniques are needed, to deal with models in which the major part of components are of discrete-time nature.

Among analyses existing in the literature, we have chosen in this paper the analysis of the basic basic G/G/1 system to illustrate the application of discrete transforms. The reason for this choice is that in this case, a broad spectrum of methods has been developed in both time and transform domains. It should be noted here that in almost all other discrete-time performance analyses, methods and algorithms invented normally operate in time or probability domain, due to the higher level of model complexity.

The most important advantage of discrete-time transforms is the ability to provide results directly in the form of probability distributions or distribution functions. These often give better insights to support performance studies compared to results given in transform domains (e.g. Laplace transforms), which we often obtained using continuous-time modeling techniques.

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