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Discrete-Time Analysis for the Interdeparture Distribution of GI/G/1 Queues

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The class of GI/G/1 queues with discrete-time distributions of arrival and service processes is considered. This type of models arises out of measurements in the form of histograms or in modelling approaches of time-slotted systems, packet switching systems, etc.. Using known discrete-time algorithms for the calculation of the waiting time distribution function, a new numerical analysis method for the idle period and the output process is presented in this paper, whereby the distribution of the interdeparture time is obtained. The method described is based on the evaluation of discrete convolution operations using fast convolution algorithms, e.g.,the Fast Fourier Transform, in conjunction with the concept of virtual unfinished work. In order to illustrate the use of the method, numerical examples are given for different types of input and service processes.

1. INTRODUCTION

In performance analysis of computer communication systems, the parameterization of model components is often based on measured data, which are given in the form of histograms. The discrete-time model elements can be, e.g., arrival or service processes. This leads to the class of discrete-time models. Models with time-discretized components are also obtained in performance investigations of packet switching systems, time-slotted systems etc..

Since most of performance evaluation methods are given in continuous-time domain, exact and approximate analytical investigations for discrete-time models are usually based on equivalent continuous-time models, e.g., the discrete-time arrival and service processes are approximately described by means of random variables with well-known time-continuous types of distribution functions.

There are a number of studies [1,2,8,16,17,18,20,30,31], which deal with analysis approaches for discrete-time models. Queueing networks with time-discretized processes are the subjects of the investigations given by Bharath-Kumar [2] and Pujolle et al. [31]. In [1,17,18,30], methods for the analysis of discrete-time queueing systems are discussed, whereby in Kobayashi [17] and Kobayashi and Konheim [18] surveys are given. Konheim [20] provided a solution of discrete-time GI/G/1 queues employing polynomial factorization, whereby the assumption of non-zero interarrival time was made. A number of papers investigated bounds and mean values of waiting time in GI/G/1 queues [11,14,15,21]. Numerical results of standard continuous-time queueing systems, including GI/G/1 queues, can be found in [22,34]. Effective algorithms for the calculation of the waiting time distribution function of the discrete-time GI/G/1 queue are presented by Ackroyd [1], where methods used in signal processing theory - in both, time and frequency domains - and fast convolution algorithms are employed.

In this paper, the class of GI/G/1 queues with discrete-time arrival and service processes is taken into account. In section 2, basic equations are derived and an outline of the method is given, which is used for the calculation of the waiting time distribution, whereby algorithms for the numerical evaluation using signal processing methods (DFT, FFT, Cepstrum,..) are briefly discussed. Based on the waiting time analysis, algorithms for the calculation of the idle period and interdeparture distribution functions are presented in section 3. Finally, numerical examples are given in section 4 for GI/G/1 queues with arbitrary chosen discrete-time processes and for systems with negative binomially distributed arrival and service times.

2. WAITING TIME DISTRIBUTION FUNCTION

Since the analysis of the idle period and the output process is based on the waiting time distribution function, a short description of the algorithm for the calculation of the waiting time distribution function used in section 3 is given in this section. Attentions are devoted to the discrete-time version of the Lindley integral equation [24] for the GI/G/1 queue [1,16]. While the waiting time distribution is derived assuming queues with first-come, first-served service discipline, the idle period and interdeparture time distributions derived in the next section are also valid for service in random order or last-come, first-served discipline.

2.1 General

In the literature, a number of analysis approaches in accordance with the calculation of the waiting time distribution function of the GI/G/1 queue can be found [6,8,10,19,20,28,30,36,38]. Most of these methods are related to solutions of the Wiener-Hopf equation in Laplace domain,which are based on techniques like spectral factorization, numerical poles and zeros allocation of the system function, determination of quadratic factor of polynomials [30], as well as separation of functions having convolutions in frequency domain. Schassberger [33] considered queues with interarrival and service times of "almost phase type". Ackroyd [1] presented an effective algorithm for the calculation of the waiting time distribution of the discrete-time GI/G/1 queue, where concepts of signal processing theory (e.g., the cepstrum concept [27], phase unwrapping technique [37], etc..) and fast convolution algorithms are used.

Although the derivation of the basic Lindley integral equation can be found in a large number of publications on queueing theory, we will outline a derivation for the special case of discrete-time processes, whereby random variables are introduced and some basic functional relationships between them are derived, which will be used for the idle period and output process analysis in the next section.

2.2 The Lindley Integral Equation in Discrete Time Domain

Fig. 1 shows a sample path of the unfinished work in a GI/G/1 queue. The following discrete time random variables (r.v.) are defined:

- A_n r.v. for the interarrival time between the n-th and the (n+1)-st arrival
- A r.v. for the equilibrium interarrival time
- B_n r.v. for the service time of the n-th arrival (n-th customer)
- B r.v. for the equilibrium service time
- U_n^- r.v. for the amount of unfinished work immediately before the arrival instant of the n-th customer

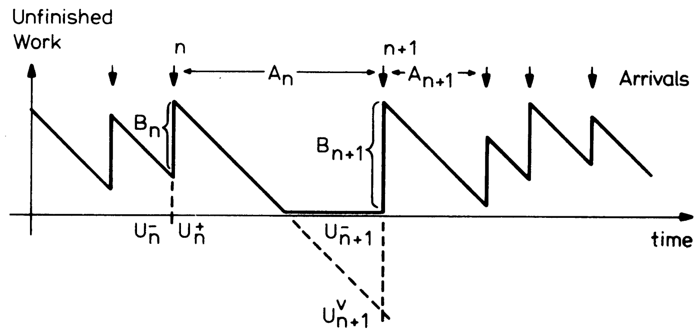

Fig.1 A sample path of unfinished work in GI/G/1 queues: definition of parameters and random variables

$W_n = U_n^-$ r.v. for the waiting time of the n-th customer

U_n^+ r.v. for the amount of unfinished work immediately after the arrival instant of the n-th customer

U_{n+1}^v r.v. for the virtual unfinished work in the system immediately before the arrival instant of the (n+1)-st customer, i.e. the amount of unfinished work in the case, in which the server would continuously serve even if the system is empty so that the unfinished work could take negative values.

We consider the random variables to be of discrete-time nature, i.e., the time axis is divided into intervals of length Δt. Thus, samples of the random variables are integer multiples of Δt, i.e., we have an equidistant time discretization. The discretization intervals can be thought of as in packet switching systems where Δt is the transmission time of a bit, byte or a whole packet, or the slot length in time-slotted communication systems. In the case of data used for the modelling approach are delivered from measurements, Δt may stand for the time resolution of the measurement monitoring system.

Further, we use the following notations for functions belonging to a r.v. X

$x(k) = \Pr\{X = k\} \qquad -\infty < k < +\infty$ distribution of X

$X(k) = \sum_{i=-\infty}^{k} x(i) \qquad -\infty < k < +\infty$ distribution function of X

EX mean of X

c_X coefficient of variation of X.

For a given distribution x(k) the Z-Transform is denoted by

$$x_{ZT}(z) = \sum_{k=-\infty}^{+\infty} x(k)\, z^{-k} \quad . \tag{2.1}$$

For a finite distribution of a positive, discrete random variable with $\{x(k), k=0,\dots,N-1\}$, the Z-Transform can be determined via the Discrete Fourier Transform (DFT) using the sampling theory [5,13,27] :

$$x_{DFT}(k) = x_{ZT}(e^{i \cdot 2\pi \cdot k/N}) \qquad , k=0,1,\dots,N-1 \tag{2.2a}$$

or

$$x_{DFT}(k) = \sum_{n=0}^{N-1} x(n) \cdot e^{-i \cdot 2\pi \cdot kn/N} \quad , k=0,1,\dots,N-1. \tag{2.2b}$$

The Discrete Fourier Transform can effectively be evaluated using algorithms like the Fast Fourier Transform (FFT), by means of which discrete convolution operations within the algorithms described in this paper are calculated.

Considering the amounts of unfinished work just before and after the arrival instant of the n-th customer we obtain

$$U_n^+ = U_n^- + B_n \tag{2.3a}$$

or

$$u_n^+(k) = u_n^-(k) * b_n(k) \tag{2.3b}$$

where the symbol "*" denotes the discrete convolution operation.

Taking into account the virtual unfinished work (c.f. Fig.1), which continuously decreases during an interarrival interval, the following equations can be found

$$U_{n+1}^v = U_n^+ - A_n \tag{2.4a}$$

or

$$u_{n+1}^v(k) = u_n^+(k) * a_n(-k). \tag{2.4b}$$

Finally, we take into consideration the fact that the amount of unfinished work remains zero when the system is empty (e.g., during an idle period) to find the relationship between the virtual and the real unfinished works

$$U_{n+1}^- = \max(0, U_{n+1}^v) \tag{2.5a}$$

or

$$u_{n+1}^-(k) = \Pi\,(u_{n+1}^v(k)) \quad , \tag{2.5b}$$

where Π denotes the operator which "sweeps the probability in the negative line up to the origin" (c.f. [1,16]) :

$$\Pi(x(k)) = \begin{cases} x(k) & k>0 \\ \sum_{i=-\infty}^{0} x(i) & k=0 \end{cases} . \tag{2.6}$$

Taking together eqns.(2.3), (2.4), (2.5) and taking into account the associative and commutative properties of the discrete convolution operation, we obtain

$$\begin{aligned} u_{n+1}^-(k) &= \Pi(u_n^-(k) * a_n(-k) * b_n(k)) \\ &= \Pi(u_n^-(k) * c(k)) \quad . \end{aligned} \tag{2.7a}$$

It should be noted here that the function

$$c(k) = a_n(-k) * b_n(k) \tag{2.7b}$$

is often thought of as the system function of the GI/G/1 queue.

Eqn. (2.7a) represents a generalized discrete form of the Lindley integral equation, where customers may have individual interarrival and service time distribution functions. Since the waiting time distribution can successively be calculated, transient behaviour of the waiting time can also be investigated. In the following, we restrict ourself to the case of independently identically distributed r.v. A and B as well as to the calculation of the equilibrium waiting time distribution function

$$w(k) = \lim_{n \to \infty} u_n^-(k) . \tag{2.8}$$

With eqns.(2.7a,b)

$$w(k) = \Pi(w(k) * a(-k) * b(k)) = \Pi(w(k) * c(k)) . \tag{2.9}$$

Eqn. (2.7a,b) for the transient case or eqn.(2.9) for the equilibrium system can be used as the basic relations for an algorithm to calculate the waiting time distribution and the corresponding waiting time distribution function.

2.3 Calculation Algorithm in Probability Domain

According to eqn.(2.7a) the waiting time distribution of the (n+1)-st customer can be calculated from the waiting time distribution of the n-th customer and the system function. Using this fact (c.f. [1,16]) the equilibrium waiting time distribution can be iteratively determined, as schematically depicted in Fig. 2. The iteration procedure can start, e.g., with the first customer finding an empty system, i.e., his waiting time distribution is of the form

$$w_1(k) = u_1^-(k) = \begin{cases} 1 & k=0 \\ 0 & \text{otherwise} \end{cases} . \tag{2.10}$$

For large vector sizes of the arrival and service distributions, the discrete convolution operation can effectively be implemented using standard algorithms, e.g., the Fast Fourier Transform (based on the Discrete Fourier Transform) [1,5,

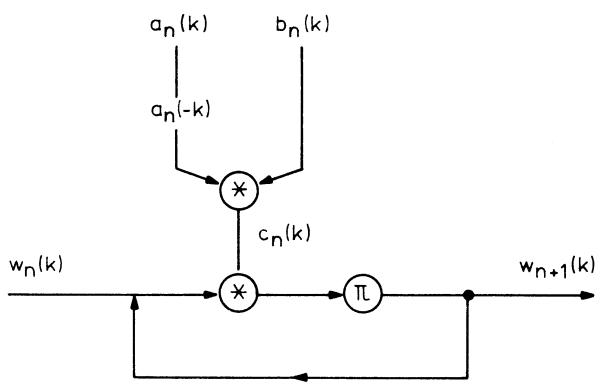

Fig. 2 *Algorithm for the waiting time distribution of GI/G/1 queues in probability domain*

13,27]. However, the number of iteration cycles required and in accordance with it, the computing efforts, depend strongly on system parameters. A more effective algorithm [1], which operates in frequency domain and uses a separation technique based on the cepstrum concept [27,37], can be applied in order to optimize the computing effectiveness.

In comparison with algorithms in transform domain, e.g.,

- the spectral factorization in Laplace- or Z-domain, which requires rational transform functions or
- the separation of maximum- and minimum-phase systems using the cepstrum concept [1,27], which fails in the case of system functions c(k) having equidistant zeros,

the algorithm in probability domain (or time domain) is very robust with respect to the type of interarrival and service processes. Furthermore, the algorithm in time domain illustrated in Fig. 2 is applicable to GI/G/1 systems with time- or state-dependent interarrival and service time distributions, e.g., systems with workload-oriented overload control or GI/G/1 queues with alternating input processes. Since the probability a(0) is allowed to exist, the analysis of systems with group arrival processes and geometrically distributed group size is implicitly included. By means of appropriate modification of the service time distribution b(k), more general group size distributions can also be dealt with.

3. OUTPUT PROCESS : INTERDEPARTURE DISTRIBUTION FUNCTION

Using the waiting time distribution according to eqn.(2.9), the equilibrium distributions of the idle period I and the interdeparture time D will be derived in this section.

3.1 General

In most of queueing systems, the output process is non-renewal and difficult to describe. There are few cases, where the output process is explicitly analyzed, i.e., both the interdeparture distribution function and the interdependence between successive interevent times, are taken into account. Burke [3,4] showed that the output process of a M/M/n queue is Poissonian. A proof of this property, using the reversibility of the Markovian stationary process, is given by Reich [32]. In Pack [29] the output process of the M/D/1 queue is extensively analyzed. Heffes [12] investigated the output of queueing stations of type GI/M/n with Interrupted Poisson Input. With exceptions of the M/M/n queue and the special cases mentioned, most of investigations are restricted to the determination of the distribution function of the interdeparture time. Marshall [26] derived expressions for the first two moments of the interdeparture distribution function. In Makino [25], moments of the output distribution function are given for systems of types M/G/1, $E_k/M/1$ and $E_2/E_2/1$. A survey on output processes can be found in Daley [7].

The general case of discrete-time GI/G/1 queues will be investigated in the following.

3.2 Idle Period Analysis

We consider again a sample path of the unfinished work in the GI/G/1 queue and the corresponding process development of the virtual unfinished work U^v, as depicted in Fig. 3. Since the service time and the amount of unfinished work are principally interchangable, the length of an existing idle period is exactly the amount of virtual unfinished work with negative values. Thus, the idle period distribution can be derived by means of the equilibrium distribution of the virtual unfinished work given by

$$u^v(k) = w(k) * a(-k) . \tag{3.1}$$

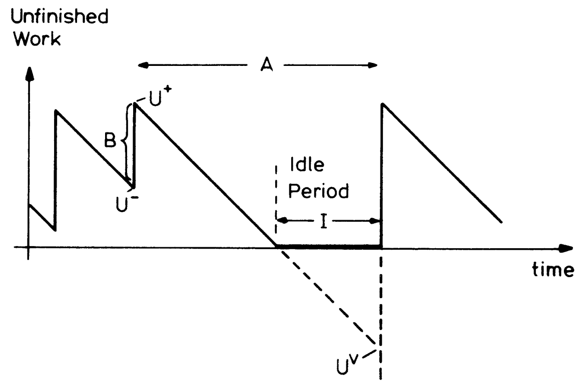

Fig. 3 *On the analysis of idle period distribution in GI/G/1 queues*

We obtain

$$i(k) = \Pr\{I = k\} = \Pr\{U^v = -k\} = K\, u^v(-k) \tag{3.2}$$

whereby the normalization constant K can be calculated as

$$K^{-1} = \sum_{j=1}^{\infty} u^v(-j) \; . \tag{3.3}$$

It should be noted here that eqn.(3.3) implies the definition of the idle period I to be a non-zero r.v. Finally, we arrive at the distribution :

$$i(k) = u^v(-k) \left[\sum_{j=1}^{\infty} u^v(-j) \right]^{-1} \quad , k > 0. \tag{3.4}$$

3.3 Algorithm for the Interdeparture Distribution

Fig. 4 illustrates a sample path of the long-term observation of the unfinished work in a GI/G/1 queue, which consists of an alternating chain of idle period (I) and busy period (S). Considering two consecutive departure instants n and (n+1), the interdeparture interval D is (c.f., Makino [25])

- the sum of an idle period and a service time, if the n-th departure instant is at the end of a busy period and
- just a service time, if the n-th departure instant is not at the end of a busy period.

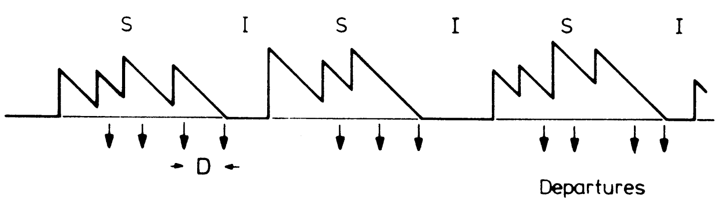

Fig. 4 *Analysis of interdeparture distribution : a sample path of unfinished work*

The probability p_E for a departure to leave an empty system is (c.f. [16])

$$p_E = \frac{EA - EB}{EI} \quad . \tag{3.5}$$

Thus, the interdeparture distribution can be given as follows

$$d(k) \;=\; p_E(i(k) * b(k)) + (1 - p_E)\, b(k), \quad k=0,1,\ldots \tag{3.6}$$

or in Z-Transform notation

$$d_{ZT}(z) = b_{ZT}(z)\;(1 - p_E + p_E\; i_{ZT}(z)) \quad . \tag{3.7}$$

Again, eqn. (3.6) or eqn.(3.7) can effectively be evaluated using algorithms for the Discrete Fourier Transform, as discussed in section 2.

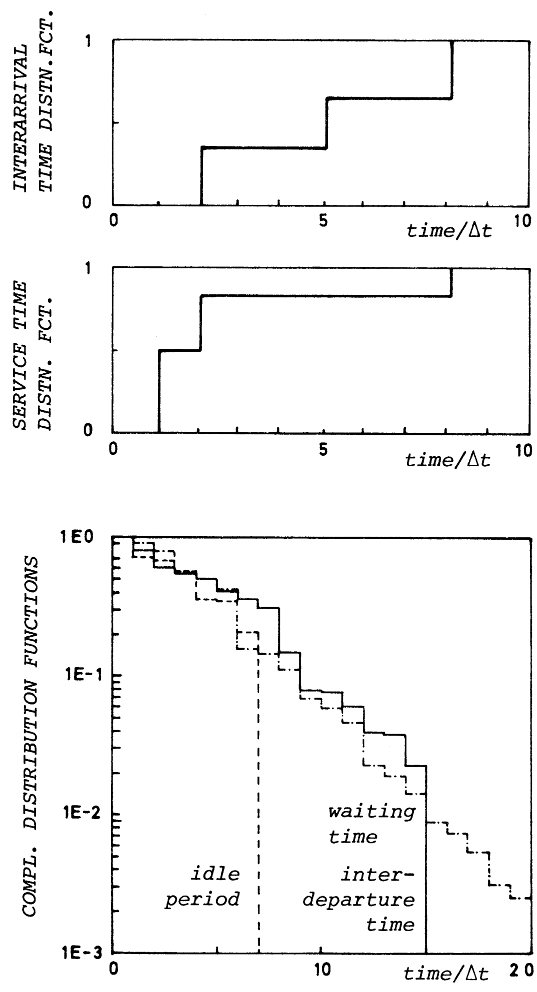

Fig. 5 *Complementary distribution functions of waiting time, idle period & interdeparture time of a discrete-time GI/G/1 queue*

4. NUMERICAL EXAMPLES

In this section, numerical examples will be presented to illustrate the calculation of the waiting time distribution and the interdeparture distribution of GI/G/1 queues. In 4.1, we consider systems with some examplary discrete-time interarrival and service time distributions. Subsection 4.2 deals with GI/G/1 queues with negative binomially distributed interarrival and service times. The normalized traffic intensity is denoted by ρ=EB/EA and the time is standardized by $\Delta t = 1$.

4.1 GI/G/1 Queues with arbitrary chosen Discrete Distributions

To illustrate the use of the method in the case of parameters arising out of measurements in the form of histograms, the following interarrival distribution a(k) and service time distribution b(k) are chosen :

$$a(2) = 25/72, \quad a(5) = 22/72, \quad a(8) = 25/72, \quad a(k) = 0 \text{ otherwise},$$

$$b(1) = 1/2, \quad b(2) = 1/3, \quad b(8) = 1/6, \quad b(k) = 0 \text{ otherwise},$$

with

$$EA = 5 \qquad EB = 2.5 \qquad \rho = 0.5$$

$$c_A = 0.5 \qquad c_B = 1.$$

Fig. 5 shows the distribution functions of the interarrival and the service times as well as the complementary distribution functions of the equilibrium waiting time (for waiting customers), the idle period and the interdeparture time.

4.2 GI/G/1 Queues with Negative Binomial Distribution for Interarrival and Service Processes

We consider in this subsection GI/G/1 queues with interarrival and service times having distributions given by their two parameters, e.g., the mean and the coefficient of variation, whereby the negative binomial distribution is employed :

$$x(k) = \binom{y+k-1}{k} p^y (1-p)^k , \qquad 0 \leq p \leq 1, \ y \ \text{ real.} \tag{4.1}$$

The mean and the coefficient of variation are given by :

$$EX = \frac{y\,(1-p)}{p} , \qquad c_X^2 = \frac{1}{y\,(1-p)} \tag{4.2}$$

or

$$p = \frac{1}{EX.c_X^2} , \qquad y = \frac{EX}{EX.c_X^2 - 1} \tag{4.3}$$

where

$$EX.c_X^2 > 1 . \tag{4.4}$$

Results for GI/G/1 queues with negative binomially distributed interarrival and service time will be discussed in the following. The queueing system is denoted by NEGBIN/NEGBIN/1. Fig. 6 shows the complementary waiting time distribution function of a NEGBIN/D/1 queue, i.e., a GI/G/1 queue with negative binomially distributed interarrival time and deterministic service time. The interarrival time coefficients of variation are chosen to be equivalent to the Erlangian of 4-th order (c_A=0.5), the Markovian (c_A=1), and the hyperexponential (c_A=1.5) distributions.

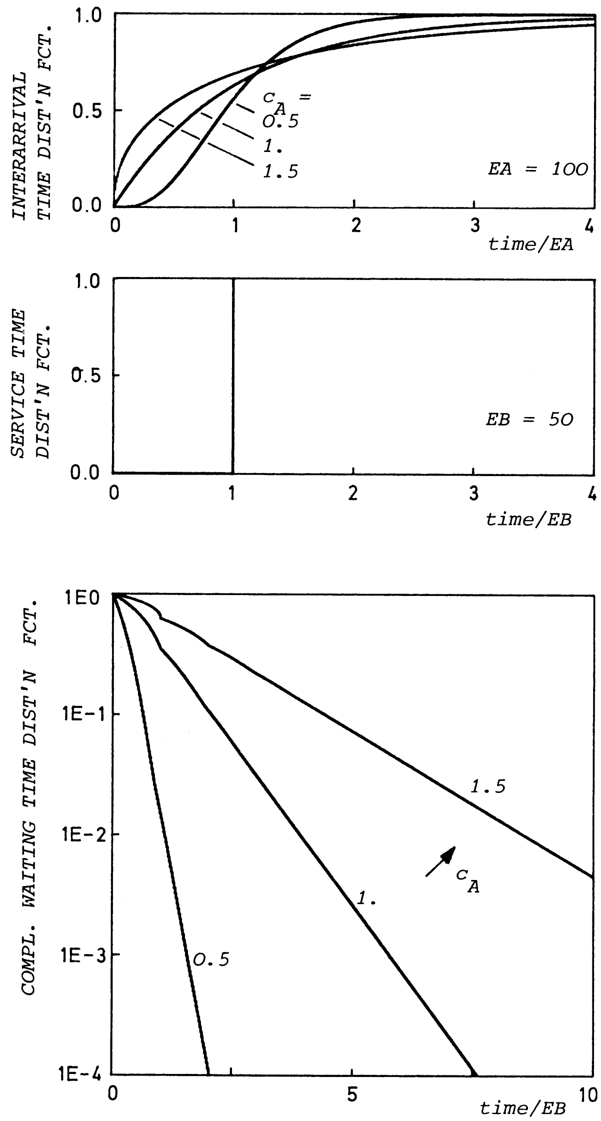

Fig. 6 Complementary waiting time distribution of a NEGBIN/D/1 queue (waiting customers)

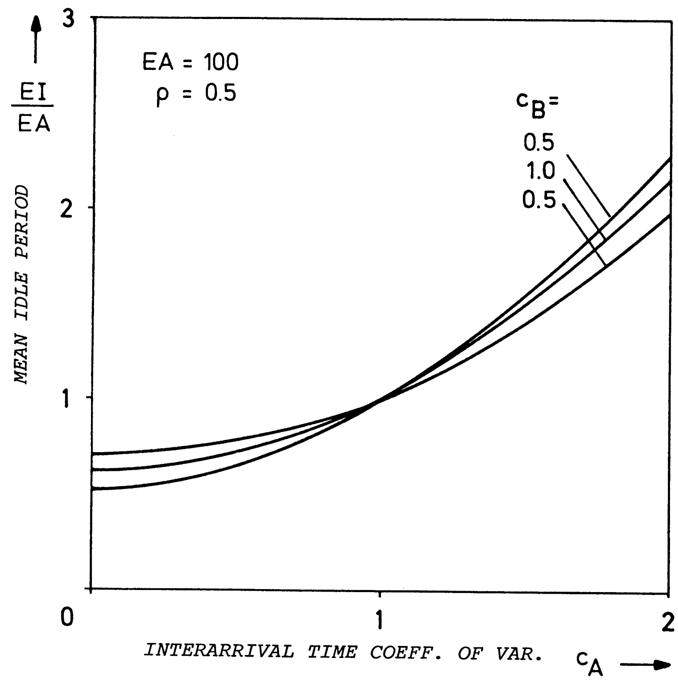

Fig. 7
Mean idle period of a GI/G/1 queue with negative binomially distributed interarrival and service times : dependency on service process

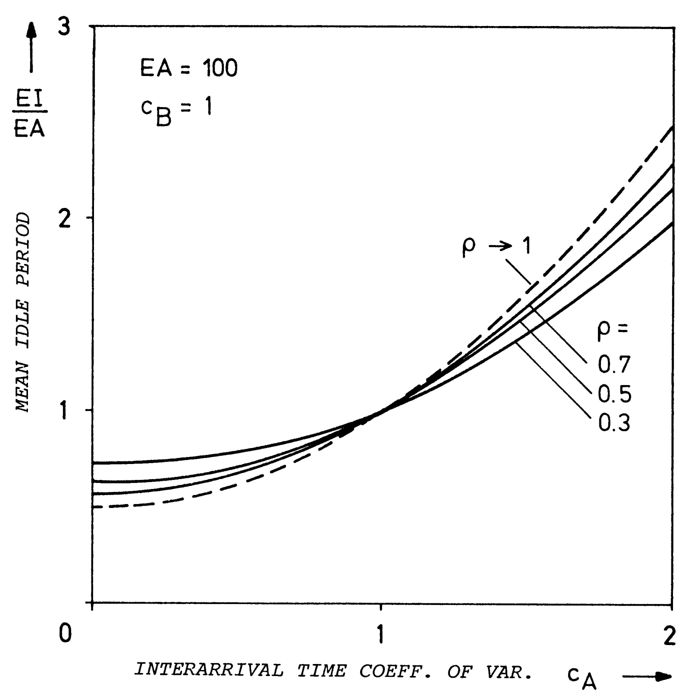

Fig. 8
Mean idle period of a GI/G/1 queue with negative binomially distributed interarrival and service times : influence of traffic intensity

The mean length of idle periods is depicted in Figs. 7 and 8 as functions of the interarrival time coefficient of variation, for different service time distributions and traffic intensities. In the case of $c_B = 1$, where the queue can be thought of as of GI/M/1 type, it can clearly be seen that the limiting curve for heavy traffic conditions ($\rho \to 1$) approaches the mean interarrival forward recurrence time, as derived by Halfin [11].

The complementary interdeparture distribution function is depicted in Fig. 9 for different values of the interarrival time coefficient of variation, which are again chosen to be equivalent to the deterministic (c_A=0), the Erlangian of 4-th order (c_A=0.5), the Markovian (c_A=1), and the hyperexponential (c_A=1.5) distributions.

The interdeparture time coefficient of variation is drawn in Fig. 10 as function of the service time coefficient of variation, for different interarrival process types and traffic intensities. In the special case of $c_A = c_B = 1$, which is equivalent to the continuous-time M/M/1 queue, we obtained a Markov-equivalent output process with $c_D = 1$, as expected.

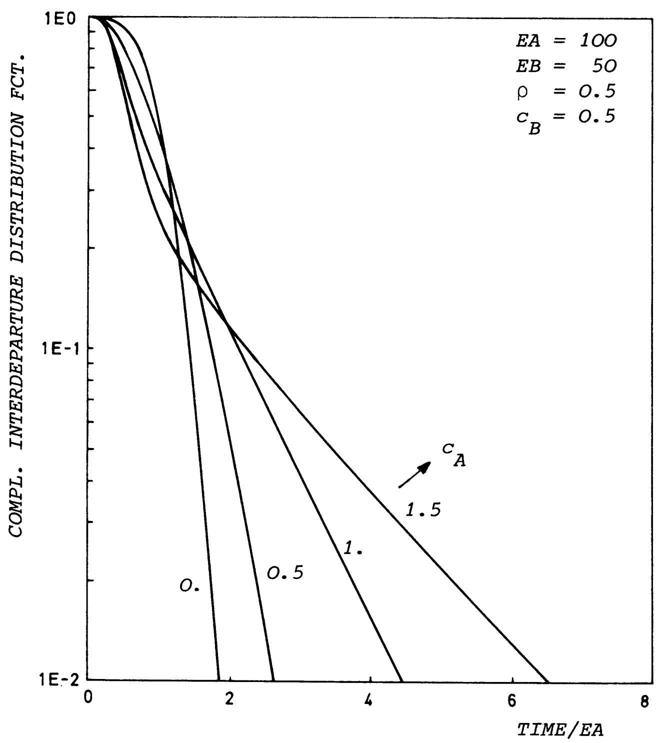

Fig. 9 *Complementary interdeparture distribution function of GI/G/1 queues with negative binomially distributed interarrival and service times*

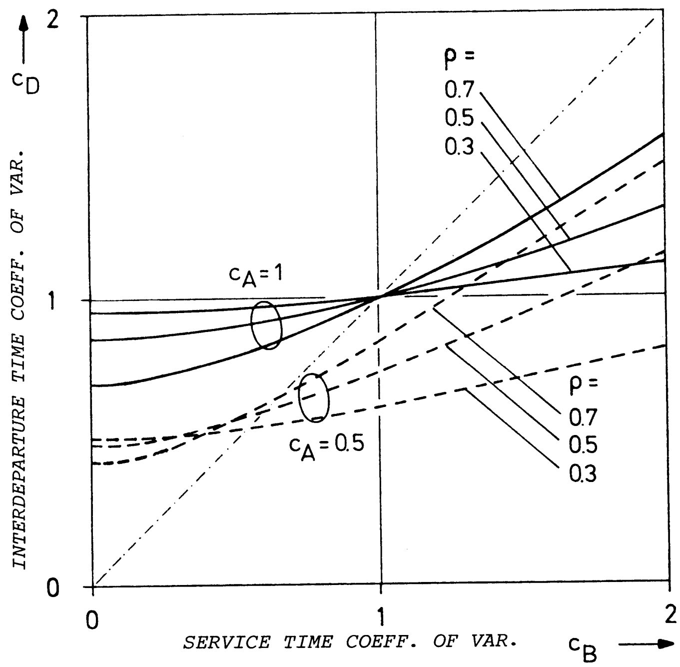

Fig. 10
Interdeparture time vs service time coeff. of variation of a GI/G/1 queue with negative binomially distributed interarrival and service times

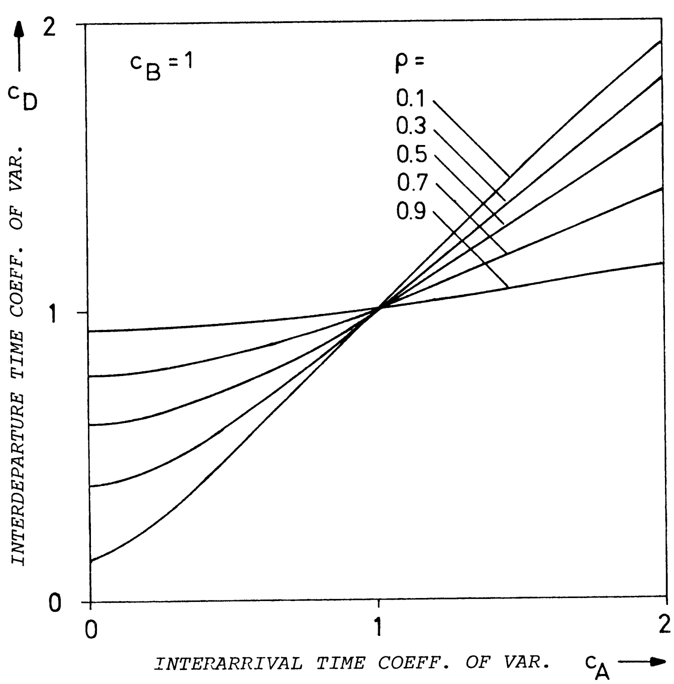

Fig. 11
Interdeparture time vs interarrival time coeff. of variation of a GI/G/1 queue with negative binomially distributed interarrival and service times: influence of traffic intensity

The influence of the traffic intensities on the output process is illustrated in Fig. 11, where the interdeparture coefficient of variation is depicted as a function of the interarrival coefficient of variation. At lower traffic intensity level, c_D approaches the value of c_A, as expected. At higher traffic intensity level, the length of idle period approaches zero, and the interdeparture process approaches the service process ($c_D=c_B$).

In order to demonstrate the accuracy of the algorithm for the interdeparture distribution, comparisons of the interdeparture time coefficients of variation are given in Tables 1 and 2. Table 1 compares the values obtained using the discrete-time algorithm for the NEGBIN/NEGBIN/1 queue (with $c_A=1$) to results for the equivalent M/G/1 queue according to Makino [25] :

$$c_D^2 = 1 - \rho^2(1 - c_B^2) \quad , \tag{4.5}$$

while Table 2 takes into account results given by Marshall [26] for the GI/G/1 queue, using the mean waiting time EW obtained in section 2 :

$$c_D^2 = c_A^2 + 2\rho^2 c_B^2 - 2\rho(1 - \rho)\frac{EW}{EB}. \tag{4.6}$$

TABLE 1 Comparison of the Interdeparture Time Coefficient of Variation of the NEGBIN/NEGBIN/1 Queue with M/G/1 Queue (c.f. Makino [25]). Parameters of NEGBIN/NEGBIN/1 Queue : $c_A=1$, EA=100.

c_D	System	$c_B=0.5$	$c_B=1.0$	$c_B=1.5$
ρ = 0.3	NEGBIN/NEGBIN/1	0.96495	0.99943	1.05431
	M/G/1 (MAKINO)	0.96566	1.0	1.05475
ρ = 0.5	NEGBIN/NEGBIN/1	0.90061	0.99943	1.14526
	M/G/1 (MAKINO)	0.90139	1.0	1.14564
ρ = 0.7	NEGBIN/NEGBIN/1	0.79463	0.99959	1.26955
	M/G/1 (MAKINO)	0.79530	1.0	1.26984

TABLE 2 Comparison of the Interdeparture Time Coefficient of Variation of the NEGBIN/NEGBIN/1 Queue with the results obtained according to Marshall [26]. Parameters of NEGBIN/NEGBIN/1 Queue : ρ=0.5, EA=100.

c_D	System	$c_B=0.5$	$c_B=1.0$	$c_B=1.5$
c_A = 0.5	NEGBIN/NEGBIN/1	0.5565010	0.7305068	0.9343127
	(MARSHALL)	0.5565011	0.7305071	0.9343137
c_A = 1.0	NEGBIN/NEGBIN/1	0.9006082	0.9994338	1.145259
	(MARSHALL)	0.9006088	0.9994346	1.145261
c_A = 1.5	NEGBIN/NEGBIN/1	1.259669	1.324368	1.429765
	(MARSHALL)	1.259670	1.324369	1.429768

5. CONCLUSION AND OUTLOOK

In this paper, an analysis method for the idle period and the interdeparture time distribution function for the class of GI/G/1 queues with discrete-time arrival and service processes is presented, whereby systems with group arrival processes can also be dealt with. This class of models can often be found in performance analysis of computer communication systems, e.g., packet switching systems, time-slotted systems etc..

Starting with an outline of the algorithm for the calculation of the waiting time distribution function, the analysis of the idle period distribution function and the interdeparture distribution function is derived. The numerical method is based on discrete convolution algorithms using the Discrete Fourier Transform (DFT), which is evaluated by means of the Fast Fourier Transform (FFT). Finally, numerical examples are given for GI/G/1 queues with arbitrary chosen discrete-time processes as well as for systems with negative binomially distributed arrival and service times.

Using the interdeparture distribution function of the GI/G/1 output processes, approximate analysis for general queueing networks consisting of GI/G/1 nodes can be done. The input-output relationship of GI/G/1 nodes forms the basic requirements for such a queueing network analysis, where decomposition arguments and renewal assumptions are taken into account (c.f.[4,9,23,32,35,39,40]). While this approach can easily and accurately be implemented in the case of feed-forward networks, the application to feedback- and closed queueing networks will additionally need iteration algorithms ; the results depend however strongly on the system parameters,e.g., the population of customers in the network. Furthermore, the algorithm for the GI/G/1 queue discussed in this paper can be modified to analyse overload control performance in switching systems, e.g., in the case of overload control strategies where the amount of unfinished work is used as overload indicator, controlling the threshold for call acceptance.

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