Contribution to the International Workshop on Applied Mathematics and Performance/Reliability Models of Computer/Communication Systems, September 1983, Pisa, Italy.

A RENEWAL APPROXIMATION FOR THE GENERALIZED SWITCHED POISSON PROCESS

P. Tran-Gia
Institute of Switching and Data Technics
University of Stuttgart
Federal Republic of Germany

In order to characterize traffic streams in distributed computer and communications systems, as well as for approximative investigations of overflow processes in telecommunication networks, the class of Switched Poisson Processes is often employed. By means of such processes, piecewise correlated input processes with high variance and state-dependent processes can be approximately modelled. This paper deals with the Generalized Switched Poisson Process SPP(G_1G_2) for which an approximation method with a renewal assumption is presented and discussed. The accuracy of the renewal approximation is shown by means of numerical results for the single server queueing system with a finite waiting capacity SPP(MM)/M/1-S.

1. INTRODUCTION AND PROCESS DESCRIPTION

In real-time processing systems with distributed control, statistical characteristics of traffic streams resulting from events interchanged between processors are very complex. On the one hand the behaviour of these streams often depends on the actual state of the system which is represented for example by the number of active processes to be scheduled in the system. On the other hand, traffic streams which result from input processes in overload situations, such as interprocess or interprocessor communications, are highly time-dependent. They must be described by means of instationary processes or approximately by means of quasistationary processes.

In order to provide model components which allow a description of such traffic streams, this paper investigates an approach of a quasi-stationary process, i.e. the generalized switched Poisson process for which a renewal approximation is presented.

There are a number of studies which utilize the regular switched Poisson process and its related marginal processes (e.g. the interrupted Poisson process) as input process of queueing models. In [8, 9] systems with interrupted Poisson input process (IPP) are discussed. While in [8] the infinite server model with IPP input is investigated, the output process of the queue GI/M/n is treated in [9]. The switched Poisson process (SPP) with Markovian phase lengths is dealt with in [7, 10, 11, 12]. In [7] the process SPP appears as a special case of the GI+M input. A solution for the delay system SPP/M/1 is given in [11], using generating functions.

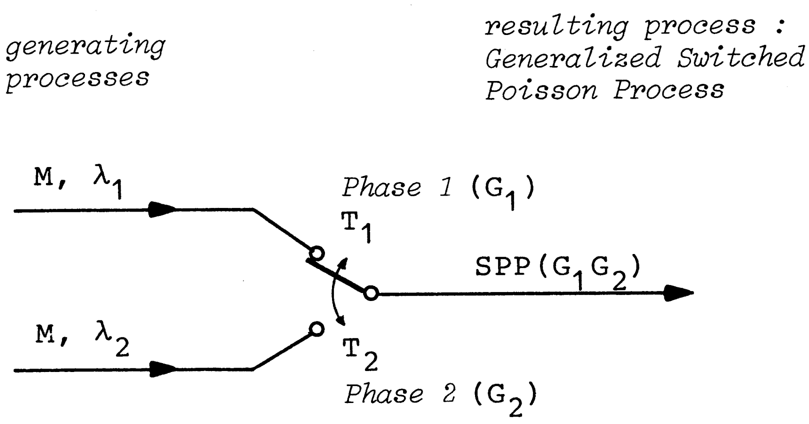

Fig. 1 *Generating model for the generalized switched Poisson process.*

A generating model of the generalized switched Poisson process is shown in Fig.1. The process results from an alternated switching between two Poisson processes (the originating processes) which are characterized by the rates λ_1 and λ_2, respectively. The visit times of the resulting process are independent and identically distributed random variables T_1 and T_2. According to the arbitrary phase length distribution functions, the following notation will be used for the generalized switched Poisson process: SPP(G_1G_2), where G_1 and G_2 denote the distribution types of T_1 (Phase 1) and T_2 (Phase 2), respectively. For $\lambda_1 = 0$, the special case of Generalized Interrupted Poisson Process IPP(G_1G_2) is defined.

Thus, the generalized switched Poisson process SPP(G_1G_2) can be completely characterized by the following random variables (r.v.):

T_1	r.v. for the length of phase 1
T_2	r.v. for the length of phase 2
T_{A1}	r.v. for the interarrival time in phase 1
T_{A2}	r.v. for the interarrival time in phase 2.

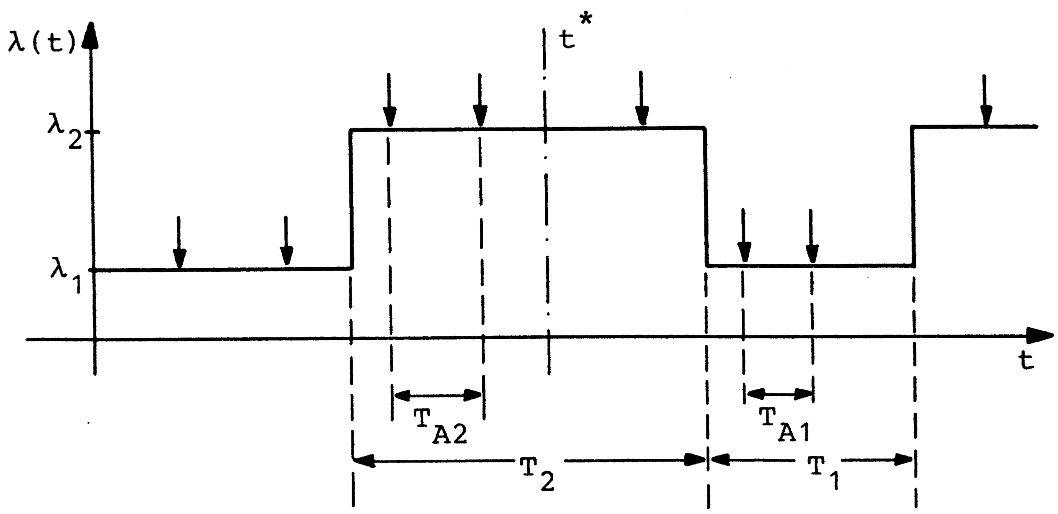

Fig. 2 *Parameters of the generalized switched Poisson process SPP(G_1G_2).*

According to the definition, T_{A1} and T_{A2} are negativ exponentially distributed r.v. corresponding to the originating Poisson processes in Fig.1, where

$$F_{A1}(t) = Pr\{T_{A1} \leq t\} = 1 - e^{-\lambda_1 t} \quad ; \; E[T_{A1}] = \frac{1}{\lambda_1} , \qquad (1.1)$$

$$F_{A2}(t) = Pr\{T_{A2} \leq t\} = 1 - e^{-\lambda_2 t} \quad ; \; E[T_{A2}] = \frac{1}{\lambda_2} . \qquad (1.2)$$

Furthermore, the mean phase lengths are denoted by:

$$E[T_1] = h_1 = \frac{1}{\omega_1} , \qquad (1.3)$$

$$E[T_2] = h_2 = \frac{1}{\omega_2} . \qquad (1.4)$$

As an alternative to the basic parameters given in eqns. (1.1 - 1.4) the following process parameters are defined for modelling purposes which allow a description of input processes (e.g. overload traffic streams) in a more realistic way:

i) The mean arrival rate (c.f. [10])

$$\lambda = \frac{\lambda_1 h_1 + \lambda_2 h_2}{h_1 + h_2} = \frac{\lambda_1 \omega_2 + \lambda_2 \omega_1}{\omega_2 + \omega_1} . \qquad (1.5)$$

ii) Considering two consecutive phases of type 1 and 2 together as a period of the process, the mean number of events in a period is

$$n_0 = \lambda_1 h_1 + \lambda_2 h_2 = \frac{\lambda_1}{\omega_1} + \frac{\lambda_2}{\omega_2} \qquad (1.6)$$

The parameters λ and n_0 can be used to characterize the switching frequency of the SPP($G_1 G_2$)-process.

iii) The ratio of phase lengths

$$\Theta = \frac{h_2}{h_1} = \frac{\omega_1}{\omega_2} . \qquad (1.7)$$

iv) The overload factor

$$\gamma = \frac{\lambda_2}{\lambda} \, . \tag{1.8}$$

In the following it is assumed that $\gamma \geq 1$ $(\lambda_2 \geq \lambda)$, i.e. λ_2 represents the higher and λ_1 the lower load level.

Two well-known renewal processes, the interrupted Poisson process (IPP, c.f. [7]) and the Poisson process can be identified as limiting cases of the switched Poisson process. These marginal processes correspond to the limiting values of the overload factor γ :

- Poisson process:

$$\lambda_1 = \lambda_2 = \lambda \rightarrow \gamma_{min} = 1 \, . \tag{1.9}$$

- Interrupted Poisson process (IPP):

$$\lambda_1 = 0 \qquad \rightarrow \gamma_{max} = \frac{\omega_1+\omega_2}{\lambda_2\omega_1} \lambda_2 = 1 + \frac{1}{\Theta} \, . \tag{1.10}$$

2. RENEWAL APPROXIMATION FOR THE GENERALIZED SWITCHED POISSON PROCESS

2.1 General

In this chapter, an interarrival distribution function for the generalized switched Poisson process SPP(G_1G_2) will be presented and investigated, whereby the renewal property is assumed. The derivation ensues from the following steps (c.f. [6]):

i) Calculation of the distribution function of the forward recurrence interarrival time.

ii) Calculation of the interarrival distribution function with a renewal approximation.

For the notation of random variables and their related functions, the following symbols will be used:

T_i random variable (r.v.), index i

$\check{T}_i$ forward recurrence time of the r.v. T_i

$F_i(t) = \Pr\{T \leq t\}$

probability distribution function (PDF) of the r.v. T_i

$f_i(t) = dF_i(t)/dt$

probability density function (pdf) of the r.v. T_i

$\Phi_i(s) = LT\{f_i(t)\} = LST\{F_i(t)\}$

Laplace-Stieltjes-Transform of the PDF $F_i(t)$ or Laplace-Transform of the pdf $f_i(t)$.

Additionally, the conditional random variables $T_i|T_i>T_j$ and $T_i|T_j>T_i$ are introduced, which have the following *unnormalized* pdf's and Laplace Transforms :

$$f_i^*(t)|_{T_i>T_j} = f_i(t)\cdot F_j(t) \quad \circ\!\!\xrightarrow{LT}\!\!\bullet \quad \Phi_i^*(s)|_{T_i>T_j}$$
$$f_i^*(t)|_{T_j>T_i} = f_i(t)\cdot(1-F_j(t)) \quad \circ\!\!\xrightarrow{LT}\!\!\bullet \quad \Phi_i^*(s)|_{T_j>T_i} \qquad \ldots(2.1)$$

The corresponding normalized forms are :

$$f_i(t)|_{T_i>T_j} = \frac{f_i^*(t)|_{T_i>T_j}}{\int_0^\infty f_i^*(t)|_{T_i>T_j}\,dt} \quad \circ\!\!\xrightarrow{LT}\!\!\bullet \quad \Phi_i(s)|_{T_i>T_j}$$
$$f_i(t)|_{T_j>T_i} = \frac{f_i^*(t)|_{T_j>T_i}}{\int_0^\infty f_i^*(t)|_{T_j>T_i}\,dt} \quad \circ\!\!\xrightarrow{LT}\!\!\bullet \quad \Phi_i(s)|_{T_j>T_i} \qquad \ldots(2.2)$$

2.2 The Forward Recurrence Time Distribution Function

The calculation of the forward recurrence time PDF $F^V(t)$ and its Laplace-Stieltjes transform $\Phi^V(s)$ is based on an observation of the process at an arbirary time epoch t^* (the observation point, see Fig.2). The probability of seeing the process in a phase 1 can be written as follows:

$$p_1 = Pr\{t^* \text{ is in a phase of type } 1\}$$
$$= \frac{h_1}{h_1+h_2} = \frac{1/\omega_1}{1/\omega_1+1/\omega_2} = \frac{\omega_2}{\omega_1+\omega_2} \qquad (2.3)$$

and analogously:

$$p_2 = Pr\{t^* \text{ is in a phase of type } 2\}$$
$$= \frac{\omega_1}{\omega_1+\omega_2} \qquad (2.4)$$

Let the observation point t^* be now in a phase 1. Conditional on this assumption, Fig.3 illustrates three examples of the forward recurrence time T^V which is the duration from t^* to the next arrival instant.

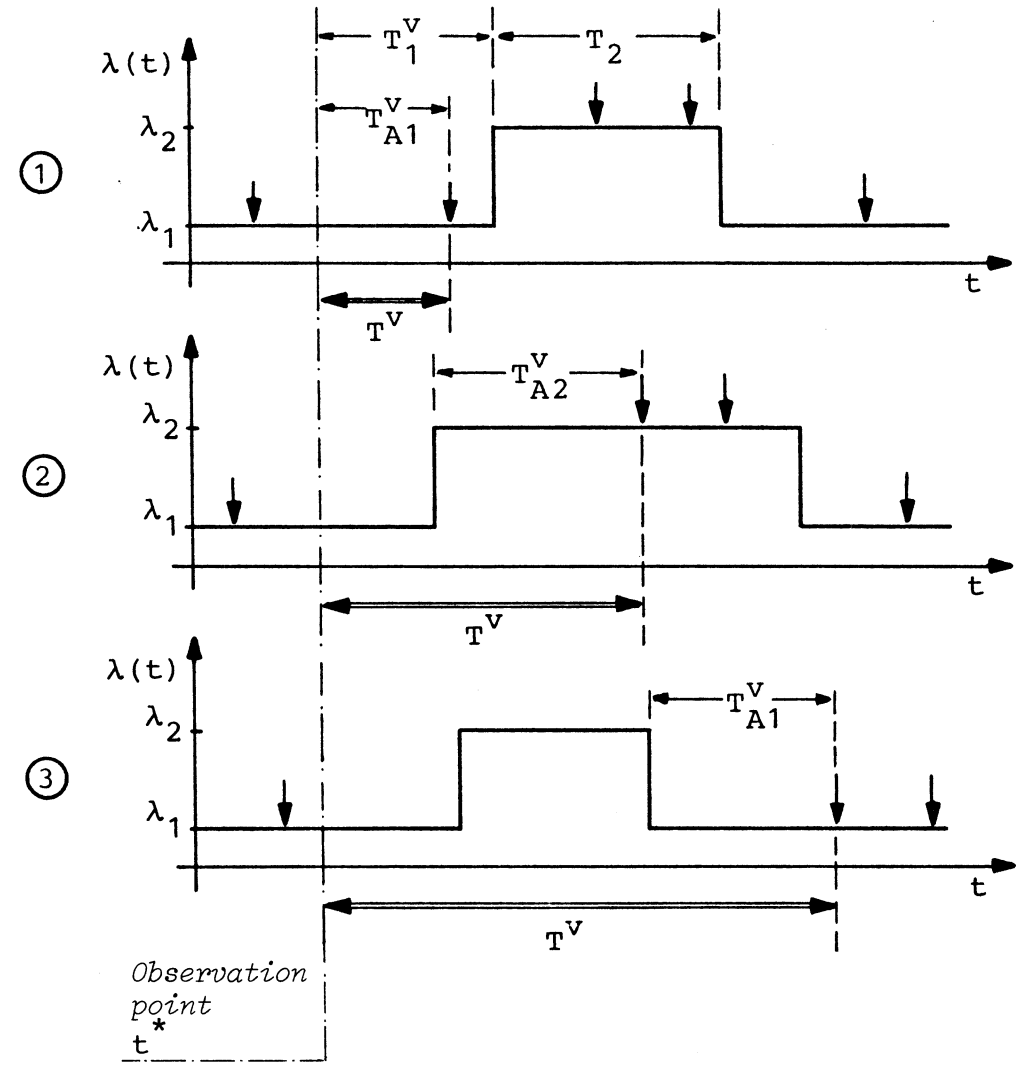

Fig. 3 *On the calculation of the forward recurrence time distribution function of the generalized switched Poisson process.*

Two cases can occur:

i) The expected event is an arrival in the current phase 1 (see case ① in Fig.3). In this case the forward recurrence time is

$$T^V = T_{A1} \mid T_1^V > T_{A1}$$

ii) The expected event is the end of the current phase 1. The process must spend the time interval $T_1^V \mid T_{A1} > T_1^V$, after which the process observation will be continued.

The process is now in the beginning of a phase 2. The following cases can occur:

i) The expected event is an arrival in the current phase 2 (see case ② in Fig.3). In this case, the compound forward recurrence time is given as

$$T^V = T_1^V | T_{A1} > T_1^V + T_{A2} | T_2 > T_{A2}.$$

ii) The expected event is the end of the current phase 2, i.e. no arrival has occured in this phase. After the phase $T_2 | T_{A2} > T_2$ the observation of the process will be continued.

The process is being observed at the beginning of a phase 1, where the following two cases can occur:

i) The expected event is an arrival in the current phase 1, (see case ③ in Fig.3). The compound forward recurrence time in this case is

$$T^V = T_1^V | T_{A1} > T_1^V + T_2 | T_{A2} > T_2 + T_{A1} | T_1 > T_{A1}.$$

ii) The end of the current phase 1 is reached and no arrival has been registered.

The observation of the process can be analogously continued until an arrival is attained. Taking into account all combinatorial possibilities for the forward recurrence time T^V, a phase diagram as shown in Fig.4 can be obtained. It should be recalled here that the observation point t^* is assumed to be in a phase of type 1.

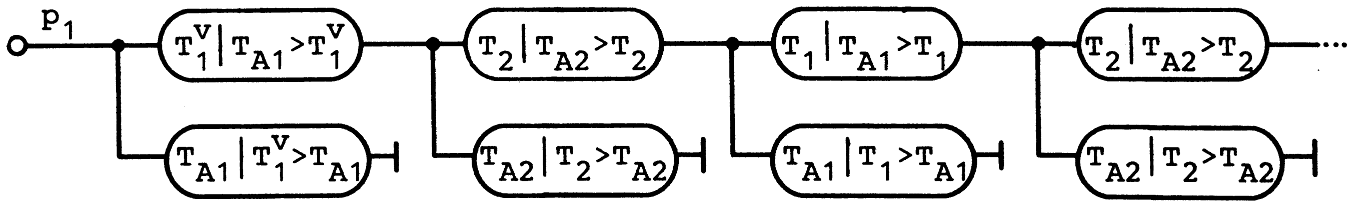

Fig. 4 *Phase diagram of the forward recurrence time (conditioned on an observation point in phase 1).*

observation point

next arrival

The combination of the two cases for the observation point t^* yields to the phase diagram of the forward recurrence interarrival time in Fig.5, where the random variables for time intervals are indicated.

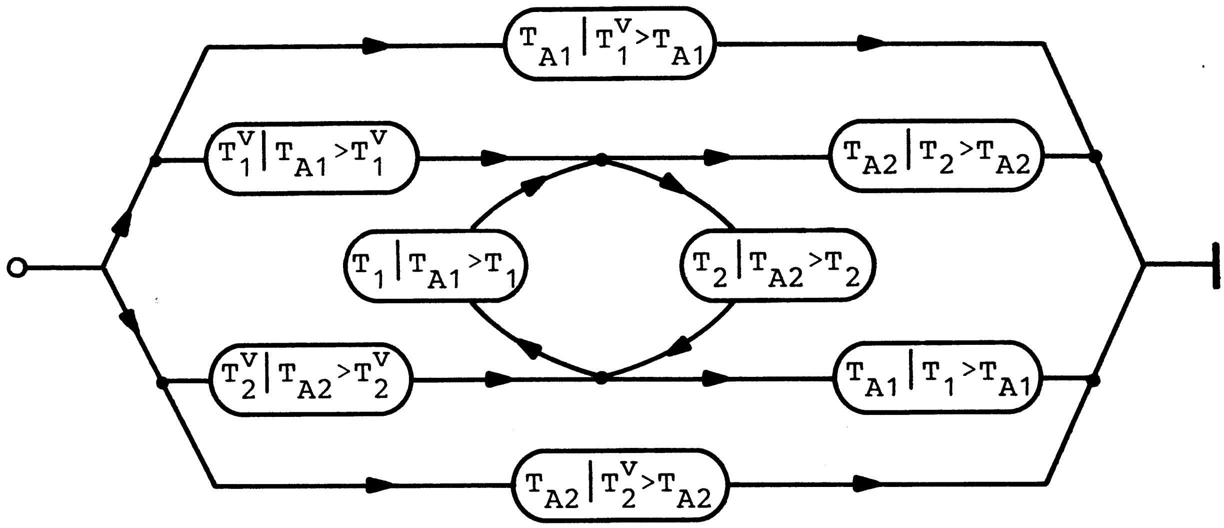

Fig. 5 *Phase diagram of the forward recurrence interarrival time*

observation point

next arrival

Considering the phase diagram in Fig.5 as a Mason flow graph [3,4], the Laplace-Stieltjes transform of the forward recurrence interarrival probability distribution function of the Generalized Switched Poisson Process SPP(G_1G_2) can be obtained as follows:

$$\begin{aligned}
\Phi^V(s) = \; & p_1 \Bigg[\Phi^*_{A1}(s)\big|_{T_1^V > T_{A1}} \\
& + \Phi_1^{V*}(s)\big|_{T_{A1} > T_1^V} \frac{\Phi^*_{A2}(s)\big|_{T_2 > T_{A2}} + \Phi^*_2(s)\big|_{T_{A2} > T_2} \cdot \Phi^*_{A1}(s)\big|_{T_1 > T_{A1}}}{1 - \Phi^*_1(s)\big|_{T_{A1} > T_1} \cdot \Phi^*_2(s)\big|_{T_{A2} > T_2}} \Bigg] \\
& + p_2 \Bigg[\Phi^*_{A2}(s)\big|_{T_2^V > T_{A2}} \\
& + \Phi_2^{V*}(s)\big|_{T_{A2} > T_2^V} \frac{\Phi^*_{A1}(s)\big|_{T_1 > T_{A1}} + \Phi^*_1(s)\big|_{T_{A1} > T_1} \cdot \Phi^*_{A2}(s)\big|_{T_2 > T_{A2}}}{1 - \Phi^*_1(s)\big|_{T_{A1} > T_1} \cdot \Phi^*_2(s)\big|_{T_{A2} > T_2}} \Bigg]
\end{aligned}$$

...(2.5)

The probabilities p_1, p_2 are given in eqns. (2.3), (2.4) and the Laplace-Stieltjes Transforms of the conditional PDFs for the conditional phases in eqn. (2.5) and Fig.5 can be calculated according to eqns. (2.1) and (2.2).

It should be recalled here that the expression for $\Phi_s^V(s)$ given in eqn.(2.5) is valid for arbitrary probability distribution functions of the phase lengths T_1 and T_2.

2.3 The Renewal Approximation

Assuming the renewal property for the generalized switched Poisson process SPP(G_1G_2), we obtain the following approximate expression for the Laplace-Stieltjes transform of the inerarrival distribution function c.f. [1, 2]):

$$\Phi(s) = 1 - \frac{s}{\lambda}\Phi^V(s). \tag{2.6}$$

where $\Phi^V(s)$ is given in eqn. (2.5).

2.4 The Special Case of Markovian Phase Lengths

As mentioned above, the expressions given in eqns.(2.5) and (2.6) can be used for arbitrary phase lengths T_1 and T_2 (c.f. Figs.1 and 2) which correspond to the notation SPP(G_1G_2). In the following we will devote attention to a special case which is often used and investigated in the literature. The phase lengths T_i (i = 1,2) are defined here to be negative exponentially distributed:

$$F_i(t) = P\{T_i \le t\} = 1 - e^{-\omega_i t}, \quad i = 1, 2. \tag{2.7}$$

Using the notation presented in chapter 1, the process is of type SPP(MM). This special case of the generalized switched Poisson process corresponds to the two-state Markov-modulated Poisson process (MMP) discussed in [10] and the input process with heterogeneous arrivals analyzed in [11].

According to the PDFs given in eqns.(1.1), (1.2), (2.7) for the r.v. T_{A1}, T_{A2}, T_1, T_2 of the process SPP(MM) the Laplace-Stieltjes transform of the conditional PDFs in eqn.(2.1) can be determined and, subsequently, the Laplace-Stieltjes transform of the forward recurrence time is calculated:

$$\Phi^V(s) = \frac{1}{\omega_1+\omega_2}\,\frac{\lambda_1\omega_2(s+\omega_1+\omega_2+\lambda_2)+\lambda_2\omega_1(s+\omega_1+\omega_2+\lambda_1)}{(s+\lambda_1)(s+\lambda_2)+\omega_1(s+\lambda_2)+\omega_2(s+\lambda_1)}. \tag{2.8}$$

Taking into account the renewal assumption in eqn.(2.6), the Laplace transform of the interarrival time of the generalized switched Poisson process with Markovian phase lengths SPP(MM) is given by:

$$\Phi(s) = \frac{1}{\lambda_1\omega_2+\lambda_2\omega_1}\cdot\frac{s(\lambda_1^2\omega_2+\lambda_2^2\omega_1)+(\lambda_1\lambda_2+\lambda_1\omega_2+\lambda_2\omega_1)(\lambda_1\omega_2+\lambda_2\omega_1)}{s^2+s(\lambda_1+\lambda_2+\omega_1+\omega_2)+\lambda_1\lambda_2+\lambda_1\omega_2+\lambda_2\omega_1}. \tag{2.9}$$

The corresponding pdf. of the interarrival time can be obtained from eqn. (2.9)

$$f(t) = LT^{-1}\{\Phi(s)\} = \frac{K}{s_2-s_1}\left((a-s_1)e^{-s_1t}+(s_2-a)e^{-s_2t}\right) \tag{2.10}$$

where

$$K = \frac{\lambda_1^2\omega_2 + \lambda_2^2\omega_1}{\lambda_1\omega_2 + \lambda_2\omega_1}, \quad a = \frac{\lambda_1\lambda_2 + \lambda_1\omega_2 + \lambda_2\omega_1}{K},$$

$$s_{1/2} = \frac{1}{2}b \pm \frac{1}{2}\sqrt{b^2 - 4aK} \quad \text{with} \quad b = \lambda_1+\lambda_2+\omega_1+\omega_2.$$

While the mean interarrival time is

$$E[T] = \frac{1}{\lambda} \tag{2.11a}$$

as expected, the coefficient of variation c can be calculated from (2.9) or (2.10):

$$c^2 = \frac{E[T^2]}{E[T]^2} - 1 = 2\,\frac{\lambda(\lambda_1+\lambda_2+\omega_1+\omega_2-\lambda)}{\lambda_1\lambda_2+\lambda_1\omega_2+\lambda_2\omega_1} - 1 \tag{2.11b}$$

As mentioned above, the formula given in eqn.(2.9) covers the whole range between the Poisson process and the interrupted Poisson process, according to the overload factor γ defined in eqn.(1.8). For these two marginal processes which have the renewal property, eqn. (2.9) can be rewritten as follows:

- Poisson process: $\lambda_1=\lambda_2=\lambda(\gamma=\gamma_{min}=1)$: $\Phi(s)=\frac{\lambda}{s+\lambda}$
- Interrupted Poisson process (IPP): $\lambda_1=0(\gamma=\gamma_{max}=1+\frac{1}{\Theta})$:

$$\Phi(s) = \frac{\lambda_2(s + \omega_1)}{s^2 + s(\omega_1 + \omega_2 + \lambda_2) + \lambda_2\omega_1} \qquad \text{(c.f. Kuczura [8])}$$

3. ON THE ACCURACY OF THE RENEWAL APPROXIMATION

In order to estimate the accuracy of the renewal approximation, we consider in this chapter the process SPP(MM) as input of a single Markovian server queueing system with finite waiting capacity S, i.e. the delay-loss system SPP(MM)/M/1-S. The mean service time $\frac{1}{\mu}$ will be used here to standardize the results.

In the following, system characteristics will be compared to validate the renewal approach and to show the dependency of the approximation accuracy on the process parameters, where:

i) The exact solution of the system SPP(MM)/M/1-S is carried out by means of a consideration of a two-dimensional Markov process. Numerical results are obtained using a recursive algorithm

ii) The renewal approximation transforms the process SPP(MM) into a general independent input process with the Laplace transform given in eqn.(2.9) or the PDF in eqn.(2.10). Thus, the results for the approximation are obtained by solving the equivalent system GI/M/1-S, using a numerical algorithm.

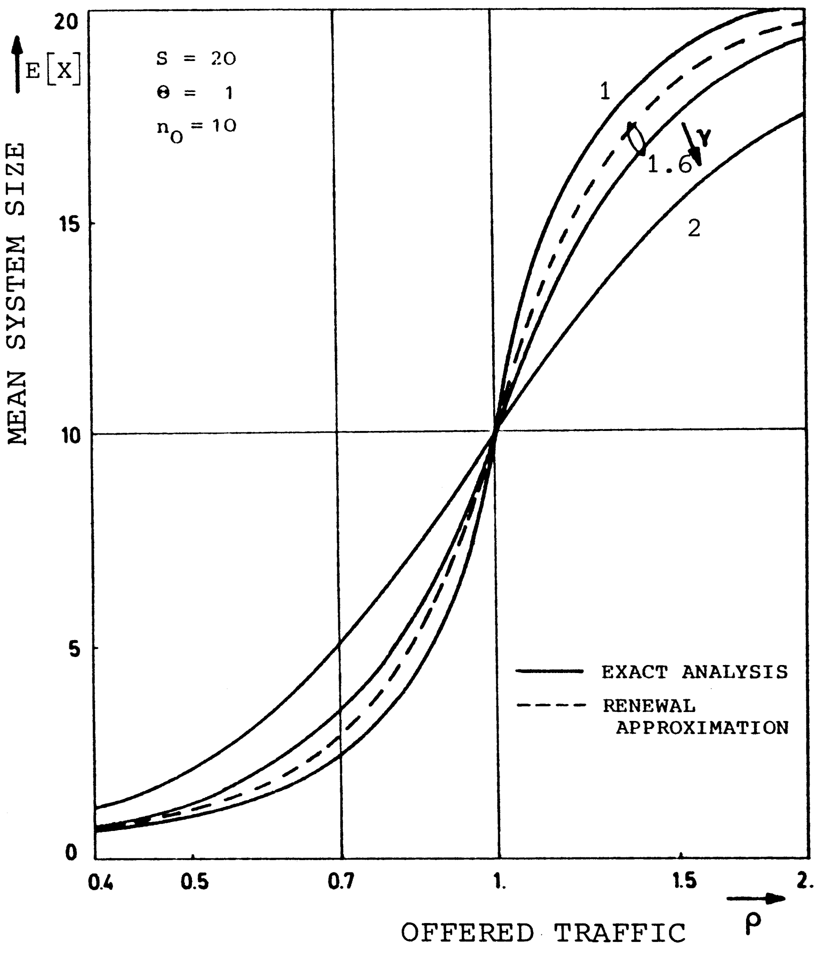

Fig. 6 *Accuracy of the renewal approximation: mean system size vs offered traffic.*

Fig. 6 shows the mean system size as a function of the offered traffic $\rho = \lambda/\mu$. For the chosen parameters (S=20, Θ=1, n_O=10) the overload factor γ can vary from γ_{min}=1 (Poisson process) to γ_{max}=1+1/Θ =2 (interrupted Poisson process). For these two marginal cases, the renewal assumption is exact as expected.

The blocking probabilities are depicted in Fig.7. For different values of the offered traffic ρ, it is seen here that the renewal assumption is a closed approximation for a wide range of γ. However, the accuracy shown here depends very strongly on the mean number n_O of arrivals per process period. This can be explained by the fact that for smaller values of n_O the switched Poisson process is more random and therefore the renewal approximation is more accurate.

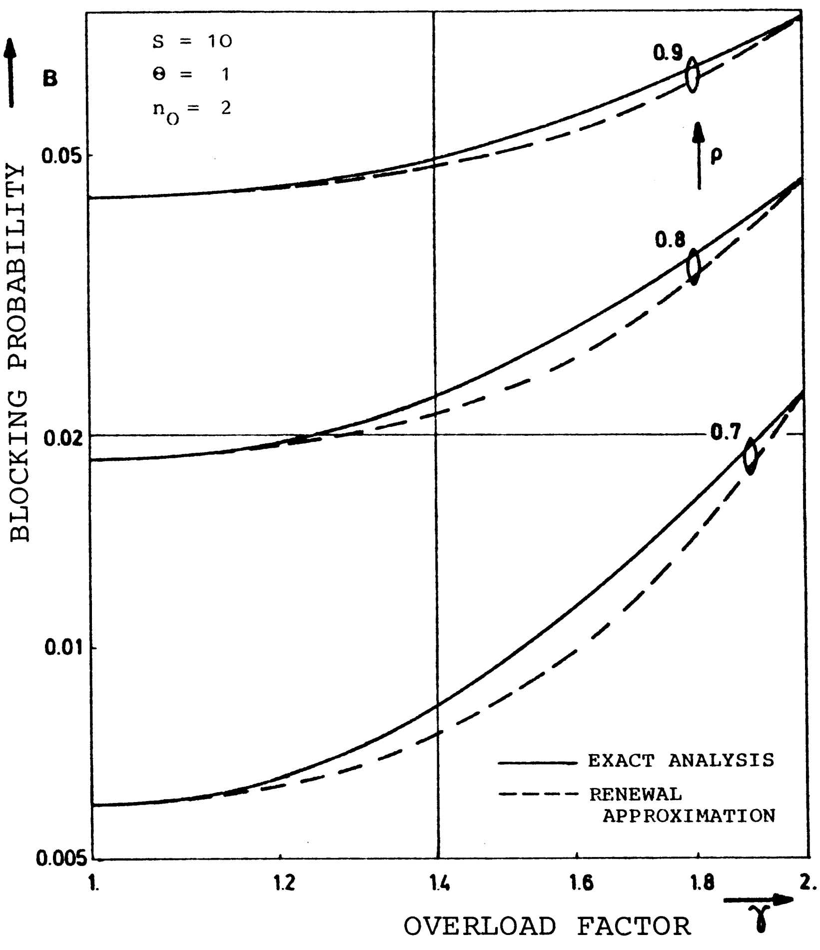

Fig. 7 Accuracy of the renewal approximation : blocking probability vs overload factor.

4. CONCLUSIONS AND OUTLOOK

In this paper an approximation method for the Generalized Switched Poisson Process SPP(G_1G_2) using a renewal assumption is developed and investigated. As shown by the reults presented for the exemplary system SPP(MM)/M/1-S the renewal approximation provided is accurate for a wide range of process parameters. The approximate formula (c.f. eqns. 2.5 and 2.6) is valid for arbitrary phase lengths of the generalized SPP. Therefore, the renewal approach represents a simplification of the analysis in the case of more complex models with SPP input, where merely the PDF of the input process or its Laplace transform is required. These models are subjects of current studies which deal with traffic streams in distributed computer and communication systems in order to estimate their performance under overload conditions. Furthermore, attention is devoted to the renewal IPP(GM) process, where more accurate approximations for overflow processes can be obtained.

ACKNOWLEDGEMENT

The author would like to thank Mr. B. Bieker and Mr. W. Fischer for fruitful discussions and for their programming efforts.

REFERENCES

[1] Cox, D.R., Renewal Theory. Methuen & Co., London (1962).

[2] Cox, D.R., Miller, H.D., The Theory of Stochastic Processes. Methuen & Co., London (1965).

[3] Mason, S.J., Feedback Theory - Some Properties of Signal Flow Graphs. Proc. of IRE, 41 (1953), 1144-1156.

[4] Mason, S.J., Feedback Theory - Further Properties of Signal Flow Graphs. Proc. of IRE, 44 (1956), 920-926.

[5] Myskja, A., A Study of Telephone Traffic Arrival Processes with Focus on Non-stationary Cases. University of Trondheim, Norwegian Institute of Technology (1981).

[6] Kuehn, P.J., Approximate Analysis of General Queuing Networks by Decomposition. IEEE Trans. Comm.,27 (1979)1, 113-126.

[7] Kuczura, A., Queues with Mixed Renewal and Poisson Inputs. Bell Syst. Tech. J. 51 (1972)6, 1305-1326.

[8] Kuczura, A., The Interrupted Poisson Process as an Overflow Process. Bell Syst. Tech. J. 52 (1973)3, 437-448.

[9] Heffes, H., On the Output of a GI/M/N Queueing System with Interrupted Poisson Input. Operations Research 24 (1976)3, 530-542.

[10] Heffes, H., A Class of Data Traffic Processes - Covariance Function Characterization and Related Queueing Results. Bell Syst. Tech. J. 59 (1980'6, 897-929.

[11] Yechiali, U., Naor, P., Queueing Problems with Heterogeneous Arrivals and Service. Operations Research 19 (1971), 722-734.

[12] Fond, S., Ross, S.M., A Heterogeneous Arrival and Service Queueing Loss Model. Naval Res. Log. Quart. 25 (1978), 483-488.