A short description of the symmetric solution set

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Given: $[A] = ([a]_{ij}) \in \mathbb{IR}^{n\times n}$ regular, $[b] = ([b]_i) \in \mathbb{IR}^n$

Interval linear system: $[A]x = [b]$.

Solution set: $\Sigma := \{\, x \in \mathbb{R}^n \mid Ax = b,\ A \in [A],\ b \in [b] \,\}$

Symmetric solution set:

$$\Sigma_{\text{sym}} := \{\, x \in \mathbb{R}^n \mid Ax = b,\ A = A^T \in [A] = [A]^T,\ b \in [b] \,\}$$

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Symmetric solution set:

$$\Sigma_{\text{sym}} \;:=\; \{\, x \in \mathbb{R}^n \,|\, Ax = b,\; A = A^T \in [A] = [A]^T,\; b \in [b] \,\}$$

Outline

1. Introduction
2. Various descriptions of Σ
3. Fourier–Motzkin elimination
4. Various descriptions of Σ_{sym}
5. Outline of a proof for a short description of Σ_{sym}

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23.12.85

Dear Prof. Rohn,

Thank you for your Christmas card. I was very sorry that you couldn't come to the interval conference in September, since I appreciate your work very much, and would have liked to exchange with you our views on linear interval equations. It is difficult to do this by writing. But let me remark on some problems.

⋮

4. Consider the problems:

(i) Find $\square\{\tilde{A}^{-1}\tilde{b} \mid \tilde{A}^T = \tilde{A} \in A, \tilde{b} \in b\}$ for symmetric A,

(ii) Find $\square\{(\tilde{A}+i\tilde{B})^{-1}(\tilde{a}+i\tilde{b}) \mid \tilde{A} \in A, \tilde{B} \in B, \tilde{a} \in a, \tilde{b} \in b\}$ for complex intervals $A+iB, a+ib$

$(\tilde{A}+i\tilde{B})^{-1}(\tilde{a}+i\tilde{b}) = \tilde{x}+i\tilde{y}$ where $\begin{pmatrix}\tilde{x}\\ \tilde{y}\end{pmatrix} = \begin{pmatrix}\tilde{A} & -\tilde{B}\\ \tilde{B} & \tilde{A}\end{pmatrix}^{-1}\begin{pmatrix}\tilde{a}\\ \tilde{b}\end{pmatrix}$

There are problems where the entries in A are dependent; so the Oettli–Prager–Beeck result doesn't apply immediately. For example if $A = \begin{pmatrix}2 & [-1,0]\\ [-1,0] & 2\end{pmatrix}$, $b = \begin{pmatrix}1.2\\ -1.2\end{pmatrix}$, then the hull $A^H b$ is $\begin{pmatrix}[0.3, 0.6]\\ [-0.6, -0.3]\end{pmatrix}$ whereas the "symmetric hull" (i) is only $\begin{pmatrix}[0.4, 0.6]\\ [-0.6, -0.4]\end{pmatrix}$.

5. Do you have results on the rectangular case? I think it is worthwhile to ~~continue~~ extend your approach to this case although it seems that there are obstacles to an immediate generalization.

With best wishes for the new Year,

yours sincerely,

Arnold Neumaier

Thus the hull inverse is not adapted to the optimal treatment of symmetric matrices.

However, at present, no special methods have been devised for this case, and we shall content ourselves with the unsymmetric treatment of symmetric matrices.

Neumaier, Interval Methods for Systems of Equations, 1990, p. 95.

Unless you are able to handle dependent data, you will never get interest of the engineers.

Babuška, conversation with J. Rohn, 1992.

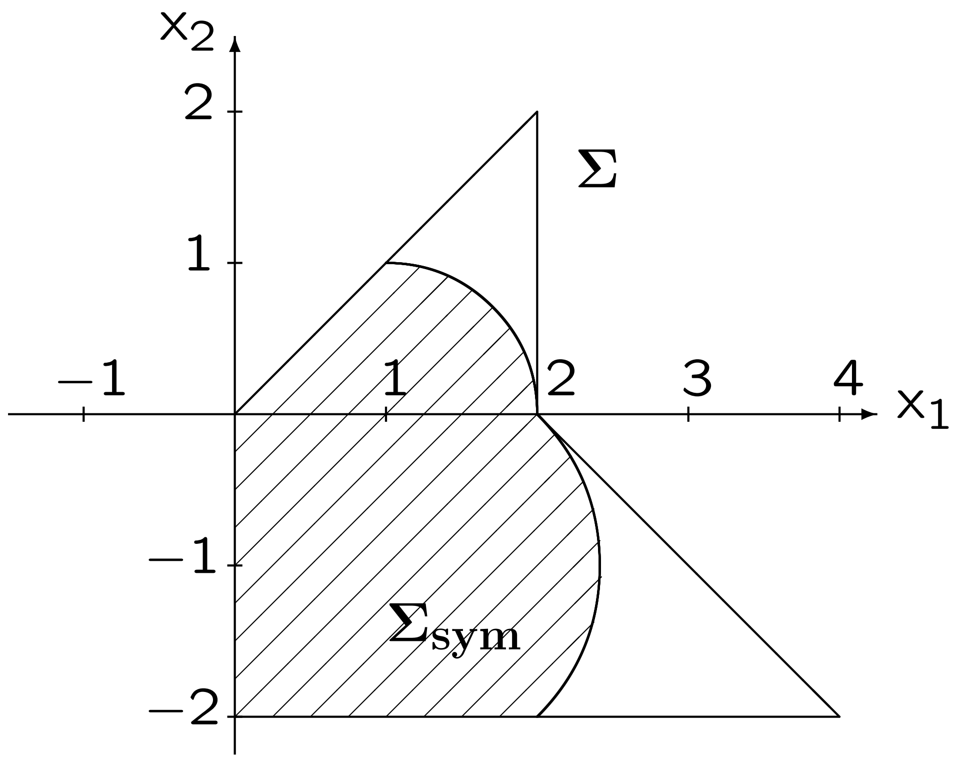
Publications on Σ_{sym}

Neumaier	1985, Dec. 23, letter to Rohn
	1990
Rohn	1986 (talk 1990; published 2004)
Jansson	1990 (talk; published 1991)
Alefeld / M.	1995 (2–dim.)
Alefeld / Kreinovich / M.	1996, 1997, 1998, 2003
M.	2001, 2012
Hladík	2008
Popova	2002, 2004, 2007

(list incomplete)

Example 1

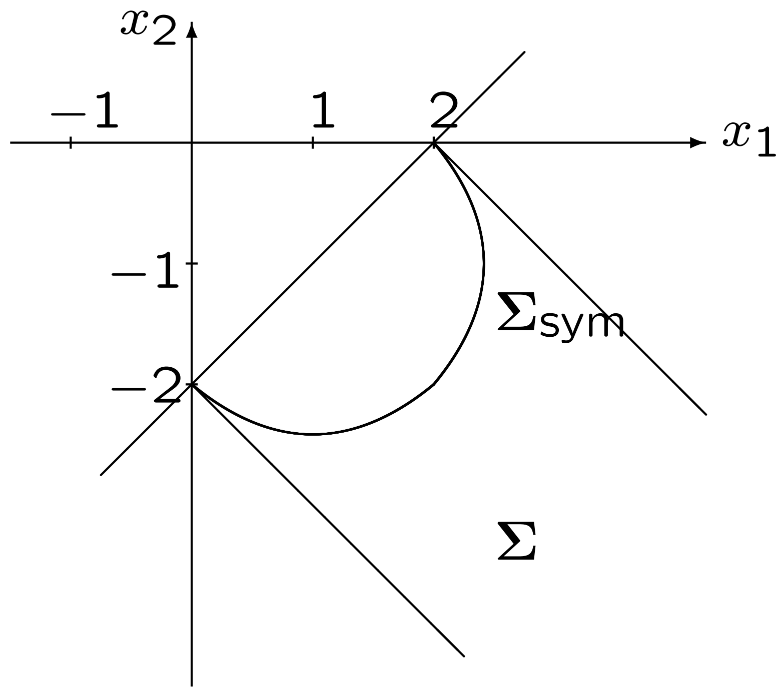
$$[A] = \begin{pmatrix} 1 & [0,1] \\ [0,1] & [-4,-1] \end{pmatrix}, \quad [b] = \begin{pmatrix} [0,2] \\ [0,2] \end{pmatrix}.$$



Example 2

$$[A] = \begin{pmatrix} 1 & [-1,1] \\ [-1,1] & -1 \end{pmatrix}, \qquad [b] = \begin{pmatrix} 2 \\ 2 \end{pmatrix}.$$

$[A]$ contains two sing. matrices but no sing. symmetric matrix.



$[-1,1] \rightsquigarrow [-1+\varepsilon, 1-\varepsilon]$: Arbitrarily large overestimation!

2. Various descriptions of Σ

$$x \in \Sigma \quad \Leftrightarrow \quad [A]x \cap [b] \neq \emptyset$$ (Beeck 1972)

$$\Leftrightarrow \quad |\check{b} - \check{A}x| \leq \mathrm{rad}([A])|x| + \mathrm{rad}([b])$$

(Oettli/Prager 1964)

$$\Leftrightarrow \quad \left\{ \begin{array}{rcl} \underline{b}_i - \sum_{j=1}^n a_{ij}^+ x_j & \leq & 0 \\ -\overline{b}_i + \sum_{j=1}^n a_{ij}^- x_j & \leq & 0 \end{array} \right\}, \quad i = 1, \ldots, n,$$

$$\text{where } [a]_{ij} =: \begin{cases} [a_{ij}^-, a_{ij}^+] & \text{if } x_j \geq 0 \\ [a_{ij}^+, a_{ij}^-] & \text{if } x_j < 0 \end{cases}$$

(Hartfiel 1980)

3. Fourier–Motzkin elimination

Start:

$x \in \Sigma_{\mathrm{sym}} \cap O_1 \qquad (O_1 \text{ first orthant})$

$$\Leftrightarrow \quad x \in O_1 \;\wedge\; \exists\, A = A^T \in [A] = [A]^T,\; b \in [b] \,:\; Ax = b$$

$$\Leftrightarrow \quad x \in O_1 \;\wedge\; \exists\, a_{ij} \in \mathbb{R} \text{ for } i,j = 1,\dots,n : \left(a_{ij} = a_{ji} \;\wedge\; \left\{ \begin{array}{c} \underline{b}_i \le \sum_{j=1}^{n} a_{ij} x_j \le \overline{b}_i \\ \underline{a}_{ij} \le a_{ij} \le \overline{a}_{ij} \end{array} \right\} \right)$$

$$\Leftrightarrow \quad x \in \Sigma \cap O_1 \;\wedge\; \exists\, a_{ij} \in \mathbb{R} \text{ for } i,j = 1,\dots,n : \left(a_{ij} = a_{ji} \;\wedge\; \left\{ \begin{array}{c} \underline{b}_i x_i \le \sum_{j=1}^{n} a_{ij} x_i x_j \le \overline{b}_i x_i \\ \underline{a}_{ij} x_i x_j \le a_{ij} x_i x_j \le \overline{a}_{ij} x_i x_j \end{array} \right\} \right)$$

Step 1: Isolation of a_{12}

$x \in \Sigma_{\text{sym}} \cap O_1$

$\Leftrightarrow x \in \Sigma \cap O_1 \;\wedge\; \exists\, a_{ij} \in \mathbb{R}$ for $i, j = 1, \ldots, n$:

$$a_{ij} = a_{ji} \wedge \left\{ \begin{array}{rcccl} \{\underline{b}_1 - \sum\limits_{\substack{j=1 \\ j\neq 2}}^{n} a_{1j}x_j\}x_1 & \leq & a_{12}x_1x_2 & \leq & \{\overline{b}_1 - \sum\limits_{\substack{j=1 \\ j\neq 2}}^{n} a_{1j}x_j\}x_1 \\ \{\underline{b}_2 - \sum\limits_{j=2}^{n} a_{2j}x_j\}x_2 & \leq & a_{12}x_1x_2 & \leq & \{\overline{b}_2 - \sum\limits_{j=2}^{n} a_{2j}x_j\}x_2 \\ \underline{a}_{12}x_1x_2 & \leq & a_{12}x_1x_2 & \leq & \overline{a}_{12}x_1x_2 \\ & & & & (1) \\ \multicolumn{5}{c}{\text{remaining } \; a_{12}\text{–free inequalities}} \end{array} \right\}$$

Step 2: Elimination of a_{12}

$x \in \Sigma_{\text{sym}} \cap O_1$

$$\Leftrightarrow \quad x \in \Sigma \cap O_1 \;\wedge\; \exists\, a_{ij} \in \mathbb{R} \text{ for } i,j = 1,\ldots,n,\ (i,j) \notin \{(1,2),(2,1)\}:$$

$$a_{ij} = a_{ji} \;\wedge \left\{ \begin{array}{c} \max\{\text{lhs}\,(1)\} \;\leq\; \min\{\text{rhs}\,(1)\} \\ \text{remaining } a_{12}\text{–free inequalities} \end{array} \right\}$$

$$\Leftrightarrow \quad x \in \Sigma \cap O_1 \;\wedge\; \exists\, a_{ij} \in \mathbb{R} \text{ for } i,j = 1,\ldots,n,\ (i,j) \notin \{(1,2),(2,1)\}:$$

$$a_{ij} = a_{ji} \;\wedge \left\{ \begin{array}{c} \underline{\text{each}}\ \text{lhs}\,(1) \;\leq\; \underline{\text{each}}\ \text{rhs}\,(1) \\ \text{remaining } a_{12}\text{–free inequalities} \end{array} \right\}$$

$$\Leftrightarrow \quad x \in \Sigma \cap O_1 \ \wedge \ \exists\, a_{ij} \in \mathbb{R} \text{ for } i,j = 1,\ldots,n,\ (i,j) \notin \{(1,2),(2,1)\}:$$

$$a_{ij} = a_{ji} \ \wedge \ \left\{ \begin{array}{c} \{\underline{b}_1 - \sum\limits_{\substack{j=1 \\ j\neq 2}}^{n} a_{1j}x_j\}x_1 \leq \overline{a}_{12}x_1x_2 \\ \{\underline{b}_2 - \sum\limits_{j=2}^{n} a_{2j}x_j\}x_2 \leq \overline{a}_{12}x_1x_2 \\ \{\underline{b}_1 - \sum\limits_{\substack{j=1 \\ j\neq 2}}^{n} a_{1j}x_j\}x_1 \leq \{\bar{b}_2 - \sum\limits_{j=2}^{n} a_{2j}x_j\}x_2 \\ \underline{a}_{12}x_1x_2 \leq \{\bar{b}_1 - \sum\limits_{\substack{j=1 \\ j\neq 2}}^{n} a_{1j}x_j\}x_1 \\ \underline{a}_{12}x_1x_2 \leq \{\bar{b}_2 - \sum\limits_{j=2}^{n} a_{2j}x_j\}x_2 \\ \{\underline{b}_2 - \sum\limits_{j=2}^{n} a_{2j}x_j\}x_2 \leq \{\bar{b}_1 - \sum\limits_{\substack{j=1 \\ j\neq 2}}^{n} a_{1j}x_j\}x_1 \\ \text{remaining } a_{12}\text{–free inequalities} \end{array} \right\}$$

4. Various descriptions of Σ_{sym}

- Dependency outside the diagonal at exactly two index pairs
- Fourier–Motzkin elimination applied to inequalities
- Fourier–Motzkin elimination applied to sets
- Oettli–Prager–like access

Theorem 1 (Hladík 2008)

Let $[A] = [A]^T \in \mathbb{IR}^{n\times n}$, $[b] \in \mathbb{IR}^n$, $x \in \mathbb{R}^n$, $r = \check{b} - \check{A}x$.
Then $x \in \Sigma_{\text{sym}}$ if and only if

$$\text{rad}([A])\,|x| + \text{rad}([b]) \;\geq\; |r| \qquad \text{(Oettli–Prager)}$$

$$\sum_{i,j=1}^{n} \text{rad}([a]_{ij})\,|x_i x_j (p_i - q_j)| + \sum_{i=1}^{n} \text{rad}([b]_i)\,|x_i(p_i + q_i)| \geq \left| \sum_{i=1}^{n} r_i x_i (p_i - q_i) \right|$$

for all $(4^n - 3^n - 2^{n+1} + 3)/2$ vectors $p, q \in \{0,1\}^n \backslash \{0, e\}$ with

$p \prec_{\text{lex}} q$ and $(\, p = e - q \;\vee\; \exists\, i : \; p_i = q_i = 0\,)$.

Theorem 2 (Thm. 1 modified and reproved; M. 2012)

Let $[A] = [A]^T \in \mathbb{IR}^{n\times n}$, $[b] \in \mathbb{IR}^n$, $x \in \mathbb{R}^n$, $r = \check{b} - \check{A}x$.
Then $x \in \Sigma_{\text{sym}}$ if and only if

$$\text{rad}([A])\,|x| + \text{rad}([b]) \;\geq\; |r| \qquad (1)$$

(Oettli–Prager)

$$|x|^T \cdot |D_p\,\text{rad}([A]) - \text{rad}([A])\,D_q| \cdot |x| \;+\; |x|^T |D_p - D_q| \text{rad}([b]) \;\geq\; |x^T (D_p - D_q) r| \qquad (2)$$

for all $(3^n - 2^{n+1} + 1)/2$ vectors $p, q \in \{0,1\}^n \backslash \{0, e\}$ with

$$p \prec_{\text{lex}} q \quad \text{and} \quad p^T q = 0, \qquad (3)$$

where $D_p = \text{diag}(p)$, $D_q = \text{diag}(q)$.

Outline of a proof for a short description of Σ_{sym}

„$\Rightarrow$“

$x \in \Sigma_{\text{sym}}$

$$\Rightarrow \quad \exists\, \tilde{A} = \check{A} + \Delta = \tilde{A}^T \in [A],\ \tilde{b} = \check{b} + \delta \in [b] :\ \tilde{A}x = \tilde{b}$$
$$\Rightarrow \quad x^T D_p \Delta x - x^T D_q \Delta^T x - x^T (D_p - D_q)\delta = x^T (D_p - D_q)(\check{b} - \check{A}x)$$

$|\cdot|$ and triangular inequality results in (2) without (3) .

„$\Leftarrow$“

Let (1) – (3) hold for some $x \in \mathbb{R}^n$.

Idea: Construct $\tilde{A} = \tilde{A}^T \in [A],\ \tilde{b} \in [b]$ s.th. $\tilde{A}x = \tilde{b}$.

Step 1: Get rid of $|\cdot|$!

a) W.l.o.g. let $x \in O_1$. $(\Rightarrow |x| = x)$

b) (3) $\Rightarrow |D_p - D_q| = D_p + D_q$

c) $|D_p\text{rad}([A]) - \text{rad}([A])D_q| = D_p\,\text{rad}([A])D_{\overline{q}} + D_{\overline{p}}\,\text{rad}([A])D_q$,

where $\overline{p} = e - p$, $\overline{q} = e - q$

d) Show (2) $\Leftrightarrow$ $\underline{L}_p^q \leq \overline{L}_q^p$ and $\underline{L}_q^p \leq \overline{L}_p^q$,

where $\underline{L}_p^q = x^T D_p(\underline{b} - \overline{A} D_{\overline{q}}\, x)$, $\overline{L}_p^q = x^T D_p(\overline{b} - \underline{A} D_{\overline{q}}\, x)$

Step 2: Apply a reverse Fourier–Motzkin elimination !

Successively for each (i,j) with $i < j$ and $x_i x_j > 0$ replace $[a]_{ij} = [a]_{ji}$ in $[A]$ by some point interval $a'_{ij} = a'_{ji} \in [a]_{ij}$ such that (1) – (3) hold for the resulting matrix.

Let $[A]^{\text{final}} = ([A]^{\text{final}})^T \subseteq [A]$ be the final matrix.

Step 3: Construct $\tilde{A} = \tilde{A}^T \in [A],\ \tilde{b} \in [b]$ s.th. $\tilde{A}x = \tilde{b}$!

(1) $\Rightarrow\ \exists\, A' = (a'_{ij}) \in [A]^{\text{final}},\ \tilde{b} \in [b]: \quad A'x = \tilde{b}$

Define $$\tilde{a}_{ij} := \begin{cases} a'_{ij} & \text{if } x_i x_j > 0 \\ a'_{ij} & \text{if } x_i = 0,\ x_j \neq 0 \\ a'_{ji} & \text{if } x_i \neq 0,\ x_j = 0 \\ a'_{ij} & \text{if } x_i = x_j = 0 \text{ and } i \le j \\ a'_{ji} & \text{if } x_i = x_j = 0 \text{ and } i > j \end{cases}$$