Metrics Choice in Interval Arithmetic

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Joint Work with Nathalie Revol

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The problem addressed in this talk

Let $f : \mathbb{R} \to \mathbb{R}$ be a real function and $x \in \mathbb{R}$ be a real number. The exact computation of $f(x)$ may be

- impossible,
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Question 1

How to compare $\mathbf{x}$ and $\mathbf{x}'$?

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Question 1

How to compare $\mathbf{x}$ and $\mathbf{x}'$?

Assume that for some real interval $\mathbf{x} \ni x$, we can compute $\mathbf{F}(\mathbf{x})$ such that $f(\mathbf{x}) \subseteq \mathbf{F}(\mathbf{x})$.

Question 2

How to measure the approximation error of $\mathbf{F}(\mathbf{x})$?

Content

How to quantify the intrinsic accuracy of an interval quantity?

How to measure the approximation error of an interval result?

Experimental results for the interval matrix multiplication

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The relative accuracy

Example

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$$\sqrt{2} \in [-2, 1.5] \quad (2)$$

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The relative accuracy

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How to quantify the quality of the enclosure?

Definition

The *relative accuracy* of an interval $\mathbf{x}$ is the quantity

$$\mathrm{racc}(\mathbf{x}) = \frac{\mathrm{rad}\,\mathbf{x}}{|\mathrm{mid}\,\mathbf{x}|}$$

By convention, $\mathrm{racc}(\mathbf{x}) = +\infty$, if $\mathrm{mid}\,\mathbf{x} = 0$.

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By convention, $\mathrm{racc}(\mathbf{x}) = +\infty$, if $\mathrm{mid}\,\mathbf{x} = 0$.

If $\mathrm{racc}(\mathbf{x}) < 1$ then $0 \notin \mathbf{x}$.
If $\mathrm{racc}(\mathbf{x}) < 1$ and $\mathbf{x} \subseteq \mathbf{y}$, then $\mathrm{racc}(\mathbf{x}) \leq \mathrm{racc}(\mathbf{y})$.

Links with other metrics

Definition (Kulpa, Markov 2003)

relative extent: $\frac{\operatorname{rad}\mathbf{x}}{\operatorname{mid}\mathbf{x}}$

Definition (Rump 1999)

relative precision: $\begin{cases} \frac{\operatorname{rad}\mathbf{x}}{|\operatorname{mid}\mathbf{x}|}, & \text{if } 0 \notin \mathbf{x} \\ 1, & \text{otherwise} \end{cases}$

Definition (Kreinovich 2013)

relative approximation error: $\min_{\tilde{x}\in[\underline{x},\overline{x}]} \max_{x\in[\underline{x},\overline{x}]} \frac{|x-\tilde{x}|}{|\tilde{x}|}$

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$\mathbf{x}$	$\mathrm{racc}(\mathbf{x})$	K. & M.	Rump	Kreinovich
$[1.414, 1.415]$	0.0007	0.0007	0.0007	0.0007
$[-2, 1.5]$	7	-7	1	1.75

Content

How to quantify the intrinsic accuracy of an interval quantity?

How to measure the approximation error of an interval result?

Experimental results for the interval matrix multiplication

The diversity of measures

Let $f : \mathbb{R} \to \mathbb{R}$ be a real function and $\mathbf{x} \in \mathbb{IR}$ a real interval.
Assume we can compute interval enclosures of f:

- $\mathbf{F}(\mathbf{x})$ such that $f(\mathbf{x}) \subseteq \mathbf{F}(\mathbf{x})$,
- $\mathbf{G}(\mathbf{x})$ such that $f(\mathbf{x}) \subseteq \mathbf{G}(\mathbf{x})$.

How to compare the computed approximates $\mathbf{F}(\mathbf{x})$ and $\mathbf{G}(\mathbf{x})$?

Different metrics found in the litterature:

absolute measures: width or radii,

relative measures: ratios of width, ratios of radii,
ratios of relative precision.

The Hausdorff distance

The set of real intervals $\mathbb{IR}$ endowed with the Hausdorff distance is a metric space.

The interval case

Let $\mathbf{x} = [\underline{x}, \overline{x}]$ and $\mathbf{y} = [\underline{y}, \overline{y}]$ be two real intervals.
The *Hausdorff distance* between $\mathbf{x}$ and $\mathbf{y}$ is the quantity

$$\begin{aligned} \mathrm{d}(\mathbf{x}, \mathbf{y}) &= \max\{|\underline{y} - \underline{x}|, |\overline{y} - \overline{x}|\} \\ &= |\mathrm{mid}\,\mathbf{y} - \mathrm{mid}\,\mathbf{x}| + |\mathrm{rad}\,\mathbf{y} - \mathrm{rad}\,\mathbf{x}|. \end{aligned}$$

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Definition

The *absolute value* (or *magnitude*) of $\mathbf{x}$ is the metric associated to the Hausdorff distance: $|\mathbf{x}| = \mathrm{d}(\mathbf{x}, 0) = |\mathrm{mid}\,\mathbf{x}| + \mathrm{rad}\,\mathbf{x}$.

Relative errors

Let $\mathbf{x} = [\underline{x}, \overline{x}]$ and $\mathbf{y} = [\underline{y}, \overline{y}]$ be two real intervals. Assume $\mathbf{x} \subseteq \mathbf{y}$.

Definition

The *relative Hausdorff error* of $\mathbf{y}$ with respect to $\mathbf{x}$ is

$$\mathrm{RHE}_{\mathbf{x}}(\mathbf{y}) = \frac{\mathrm{d}(\mathbf{x}, \mathbf{y})}{\mathrm{d}(\mathbf{x}, 0)} = \frac{|\mathrm{mid}\,\mathbf{y} - \mathrm{mid}\,\mathbf{x}| + \mathrm{rad}\,\mathbf{y} - \mathrm{rad}\,\mathbf{x}}{|\mathrm{mid}\,\mathbf{x}| + \mathrm{rad}\,\mathbf{x}}.$$

Definition

The *relative radius error* of $\mathbf{y}$ with respect to $\mathbf{x}$ is defined as the quantity

$$\mathrm{RRE}_{\mathbf{x}}(\mathbf{y}) = \frac{\mathrm{rad}\,\mathbf{y} - \mathrm{rad}\,\mathbf{x}}{\mathrm{rad}\,\mathbf{x}}.$$

The special case of interval arithmetic computations

Proposition

If $\mathbf{F} : \mathbb{IR} \to \mathbb{IR}$ is an interval enclosure of f that verifies the inclusion property, then

$$\mathrm{RHE}_{f(\mathbf{x})}(\mathbf{F}(\mathbf{x})) \leq 2\,\mathrm{RRE}_{f(\mathbf{x})}(\mathbf{F}(\mathbf{x}))$$

If $\mathbf{x} \subseteq \mathbf{y}$, then $|\mathrm{mid}\,\mathbf{y} - \mathrm{mid}\,\mathbf{x}| \leq \mathrm{rad}\,\mathbf{y} - \mathrm{rad}\,\mathbf{x}$.

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- $RHE_{[1.414,1.415]}([-2, 1.5]) = \frac{1.4145-(-0.25)+1.75-0.0005}{1.4145+0.0005} \approx 2.4$
- $RRE_{[1.414,1.415]}([-2, 1.5]) = \frac{1.75-0.0005}{0.0005} = 3499$

Content

Interval matrices: semantics

An interval matrix $\mathbf{A} \in \mathbb{IR}^{m\times n}$ is a pair of real matrices.

$$\mathbf{A} = [\underline{A}, \overline{A}] = \{M \in \mathbb{R}^{m\times n} \mid \underline{A} \leq M \leq \overline{A}\}.$$

Let $\mathbf{A} \in \mathbb{IR}^{m\times p}$ and $\mathbf{B} \in \mathbb{IR}^{p\times n}$.

$$\mathbf{AB} = \{M = AB \in \mathbb{R}^{m\times n} \mid A \in \mathbf{A}, B \in \mathbf{B}\}.$$

Interval matrix multiplication

Rump, *Fast Interval Matrix Multiplication*, 2011

Input: $\mathbf{A} = \langle M_{\mathbf{A}}, R_{\mathbf{A}} \rangle \in \mathbb{IF}^{m\times k}, \mathbf{B} = \langle M_{\mathbf{B}}, R_{\mathbf{B}} \rangle \in \mathbb{IF}^{k\times n}$

Output: $\mathbf{C} \supseteq \mathbf{AB}$

$M_{\mathbf{C}} \leftarrow fl_{\square}(M_{\mathbf{A}} M_{\mathbf{B}})$

$R'_{\mathbf{B}} \leftarrow fl_{\Delta}((k+2)\mathtt{u}|M_{\mathbf{B}}| + R_{\mathbf{B}})$

$R_{\mathbf{C}} \leftarrow fl_{\Delta}(|M_{\mathbf{A}}|R'_{\mathbf{B}} + R_{\mathbf{A}}(|M_{\mathbf{B}}| + R_{\mathbf{B}}) + \mathrm{realmin})$

return $\langle M_{\mathbf{C}}, R_{\mathbf{C}} \rangle$

where

- $\mathtt{u}$ is the roundoff unit,
- $\mathrm{realmin}$ is the smallest positive normal floating-point number,
- $fl_{\square}$: floating-point computation with rounding to nearest,
- fl_{Δ}: floating-point computation with rounding toward $+\infty$.

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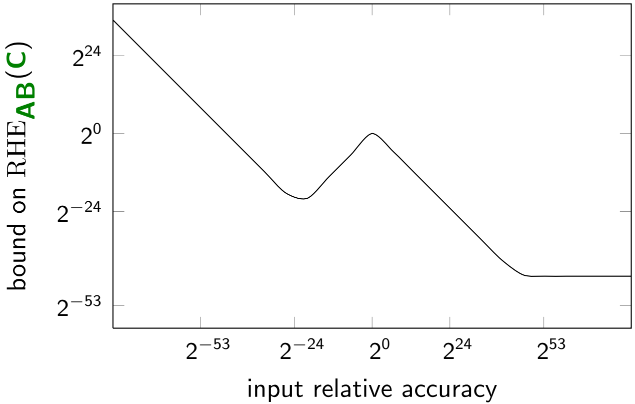
Question

How to bound overestimation error and roundoff errors?

Theoretical bound on the error

Products of 128-by-128 interval matrices in double precision



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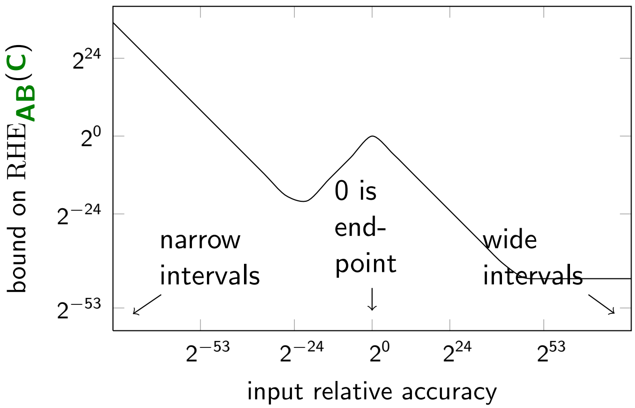

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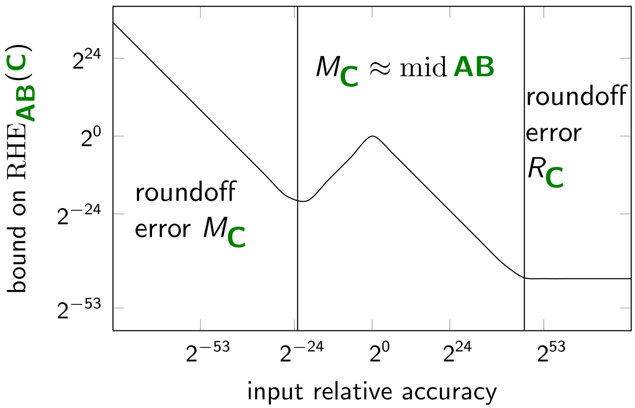
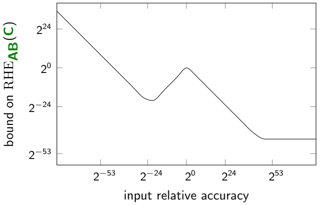


Experimental setup



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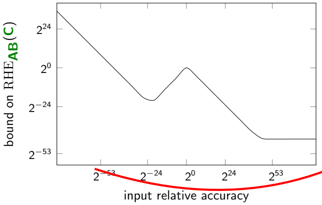

1. Take random midpoint matrices $M_{\mathbf{A}}$ and $M_{\mathbf{B}}$
2. Choose a value e for the relative accuracy and set

$$R_{\mathbf{A}} = e\,|M_{\mathbf{A}}|$$
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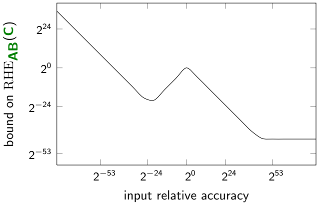

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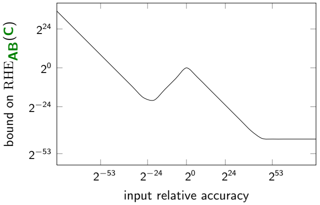

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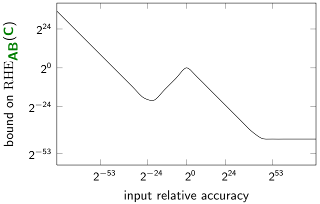

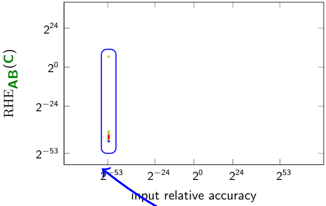

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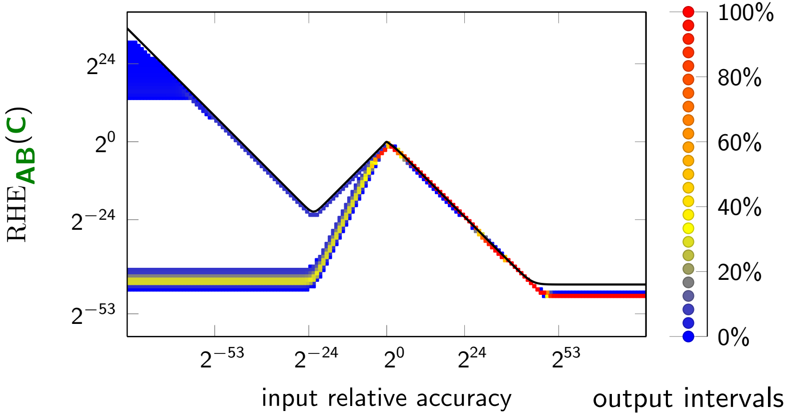
5. Report the mesured $\mathrm{RHE}_{\mathbf{AB}_{ij}}(\mathbf{C}_{ij})$

Experimental results



Conclusion

- For measuring the intrinsic accuracy of $\mathbf{x}$

$$\operatorname{racc}(\mathbf{x}) = \frac{\operatorname{rad} \mathbf{x}}{|\operatorname{mid} \mathbf{x}|}.$$

- For approximation error between intervals:
  - the relative Hausdorff error $\mathrm{RHE}_{\mathbf{x}}(\mathbf{y})$ takes midpoints and radii into account,
  - the relative radius error $\mathrm{RRE}_{\mathbf{x}}(\mathbf{y})$ is easier to bound,
  - $\mathrm{RHE}_{f(\mathbf{x})}(\mathbf{F}(\mathbf{x})) \leq 2\,\mathrm{RRE}_{f(\mathbf{x})}(\mathbf{F}(\mathbf{x}))$ when $\mathbf{F} : \mathbb{IR} \to \mathbb{IR}$ is an interval enclosure of f that verifies the **inclusion property**.

Question

Shall we come to a general agreement on the way to measure errors on interval quantities?