A numerical verification method for a basin of a limit cycle

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Introduction

- We treat dynamics of ODE's.

Problem : Autonomous system of ODE

$$\frac{d\mathbf{u}}{dt} = \mathbf{f}(\mathbf{u}), \qquad 0 < t < \infty,$$
$$\mathbf{u} \in D \subset \mathbb{R}^n,$$
$$\mathbf{f} : D \mapsto \mathbb{R}^n, \mathbf{f} \in C^\infty.$$

- Poincaré map is used for analysis of closed orbits.
- In this talk, we show how to identify a basin of an asymptotically stable closed orbit by numerical verification without defining a Poincaré map.

Verification methods for existence of closed orbits

There are several verification methods for existence of closed orbits.
Among them, we will show outlines of following 2 methods.

- Zigliczyński's method
- 2-point boundary value problems with a bordering condition

Zigliczyński's method

- Zigliczyński developed a verification method for existence of closed orbits[3].

$\Rightarrow$ This needs a narrow interval for return times.

1. Let $\mathbf{u}$ and $P(\mathbf{u})$ be an interval vector and its image of a Poincaré map, respectively.
2. $N(\mathbf{u})$ is defined as a Newton operator for the fixed point equation $P(\mathbf{u}) = \mathbf{u}$.
3. For an interval vector $[U]$, if $N([U]) \subset [U]$, then $[U]$ includes a fixed point of the Poincaré map, where $N([U]) = \{N(\mathbf{u}) | \mathbf{u} \in [U]\}$.

Features of Zigliczyński's method

- Use a Poincaré map and its variational equation.
- A variational equation can be computed with a Poincaré map simultaneously (C^1-Lohner method).
- Verification is carried out through Alefeld's theorem[3].

2-point boundary value problems with a bordering condition

- In the paper [1], we had developed a method which does not need a rigorous return time. We refer to the method as HY method.

⇒ In actual computation, we use a simplified version pointed out by Professor Matsuo and his student Kaigaishi.
Note that it is equivalent to one of 2-point boundary value problems with a bordering condition.

2-point boundary value problems with a bordering condition

HY method

- Define a Poincaré section Γ and an orthogonal projection P_Γ to Γ.
- Use Newton method with validated computation to specify a time period T^*.

$$\Updownarrow$$

Simplified method

- 2-point boundary value problems with a bordering condition.

Verification is carried out through Brouwer's fix point theorem.

Numerical verification for existence of closed orbits

We verify existence of closed orbits by 2-point boundary value problems with a bordering condition.

Notation

- $\tilde{\mathbf{u}}(t)$: an approximate periodic solution.
- $\tilde{\mathbf{w}} := \tilde{\mathbf{u}}(0)$.
- Γ : a plane which includes $\tilde{\mathbf{w}}$.
- $\mathbf{n}_\Gamma$: an unit normal vector of Γ.

For simplicity, translate the axes such that $\tilde{\mathbf{w}}$ should be the origin of new axes.

Numerical verification for existence of closed orbits

2-point boundary value problems with bordering condition

$$\begin{cases} \mathbf{n}_\Gamma^T \mathbf{w}^* = 0, \\ \mathbf{w}^* = \varphi(T^*, \mathbf{w}^*). \end{cases}$$

- $\mathbf{w}^* \in \Gamma$ is a point on the closed orbit, and T^* is the time period.
- $\varphi(t, \mathbf{w})$ indicates a point at time t on a trajectory starting at an initial point $\mathbf{w}$.

We define a mapping K to find T^* and $\mathbf{w}^*$.

Mapping K

$$\mathbf{z} = \begin{pmatrix} T \\ \mathbf{w} \end{pmatrix},$$

$$K(\mathbf{z}) = \begin{pmatrix} \mathbf{n}_\Gamma^T \mathbf{w} \\ \varphi(T, \mathbf{w}) - \mathbf{w} \end{pmatrix}.$$

Numerical verification for existence of closed orbits

- To get a zero point of $K(\mathbf{z})$, we apply the interval Newton method or the Krawczyk method for

$$K(\mathbf{z}) = 0.$$

In order to use Brouwer's fixed point theorem, we check $N([Z]) \subset [Z]$ for an interval vector $[Z]$. Here N is the interval Newton operator or the Krawczyk operator.

Numerical verification for existence of closed orbits

- If $N([Z]) \subset [Z]$, then each element of the interval $[Z] = ([T], [W])^T$ satisfies $T^* \in [T]$ and $\mathbf{w}^* \in [W]$.
- Let a ball B_η be an obtained set on Γ that is

$$B_\eta = \{\mathbf{w} \in \Gamma \mid \|\mathbf{w}\|_2 \leq \eta\},$$

and

$$[W] \subset B_\eta.$$

Note that the origin is the approximate fixed point $\tilde{\mathbf{w}}$

Identification of basin

The goal of this section is to show a sufficient condition for a given set W_α on Γ to be included by a basin of the closed orbit, where

$$W_\alpha = \{\mathbf{w} \in \Gamma \mid \|\mathbf{w} - \mathbf{w}^*\|_2 \leq \alpha\}.$$

- The center is $\mathbf{w}^*$.
- The radius is a given constant α.

Theorem

- Assumption 1: There is a family of intervals $[T_w]$ for all $\mathbf{w} \in W_\alpha$ such that

$$\varphi(T_w, \mathbf{w}) \in \Gamma,$$

where $T_w \in [T_w]$ $(T_w > 0)$. In other words, penetration condition;

$$\left(\mathbf{n}_\Gamma^T \varphi\left(\underline{T_w}, \mathbf{w}\right)\right)\left(\mathbf{n}_\Gamma^T \varphi\left(\overline{T_w}, \mathbf{w}\right)\right) < 0$$

holds when $[T_w] = \left[\, \underline{T_w}, \overline{T_w} \,\right]$.

Theorem

- $\mathbf{n}_\Gamma$ is given by $\mathbf{f}(\tilde{\mathbf{w}})/\|\mathbf{f}(\tilde{\mathbf{w}})\|_2$, which is along the flow.
- To restrict our problem on Γ, we define an orthogonal projection P_Γ as

$$P_\Gamma = I - \mathbf{n}_\Gamma {\mathbf{n}_\Gamma}^T.$$

- Assumption 2: There is $\mu (0 < \mu < 1)$ such that

$$\|\mathbf{n}_\Gamma - \frac{\mathbf{f}(\varphi(T, \mathbf{w}))}{\|\mathbf{f}(\varphi(T, \mathbf{w}))\|_2}\|_2 + \|\frac{\partial \varphi}{\partial \mathbf{w}}(T, \mathbf{w}) P_\Gamma\|_2 \leq \mu$$

holds for any $\mathbf{w} \in W_\alpha$ and any $T \in \bigcup_{w \in W_\alpha} [T_w]$.

Theorem

If Assumption 1 and 2 are satisfied, there is a sequence $(T_i, \mathbf{w}_i)$ where $T_i \in [T_{w_i}]$ and $\mathbf{w}_i \in W_\alpha$ such that

$$\mathbf{w}_i = \varphi(T_{i-1}, \mathbf{w}_{i-1}), \qquad i = 1, 2, 3, \ldots,$$

holds for each $\mathbf{w}_0 \in W_\alpha$. Moreover, the time T_i is uniquely specified within $[T_{w_i}]$, and

$$\lim_{i \to \infty} \|\mathbf{w}^* - \mathbf{w}_i\|_2 = 0,$$
$$\lim_{i \to \infty} |kT^* - T_i| = 0.$$

Here k is a certain integer corresponding to k period.

Outline of proof

For simplicity, we introduce $\mathbf{w}$, $\mathbf{w}'$, and T by

- $\mathbf{w} = \mathbf{w}_k - \mathbf{w}^*$,
- $\mathbf{w}' = \mathbf{w}_{k+1} - \mathbf{w}^*$,
- $T = T_k - T^*$,

then

$$\begin{pmatrix} T \\ \mathbf{w}' \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ \mathbf{f} & \dfrac{\partial \varphi}{\partial \mathbf{w}} \end{pmatrix} \begin{pmatrix} T \\ \mathbf{w} \end{pmatrix}.$$

Outline of proof

Multiplying certain matrices, the above becomes

$$T\mathbf{n}_\Gamma + \mathbf{w}' = \left(\left(\mathbf{n}_\Gamma - \frac{\mathbf{f}}{\|\mathbf{f}\|_2} \right) \mathbf{n}_\Gamma^T + \frac{\partial \varphi}{\partial \mathbf{w}} P_\Gamma \right) (T\mathbf{n}_\Gamma + \mathbf{w}),$$

and using Assumption 2

$$\begin{aligned} \|T\mathbf{n}_\Gamma + \mathbf{w}'\|_2 &= \|\left(\mathbf{n}_\Gamma - \frac{\mathbf{f}}{\|\mathbf{f}\|_2} \right) \mathbf{n}_\Gamma^T + \frac{\partial \varphi}{\partial \mathbf{w}} P_\Gamma\|_2 \ \|T\mathbf{n}_\Gamma + \mathbf{w}\|_2 \\ &\le \mu \|T\mathbf{n}_\Gamma + \mathbf{w}\|_2, \qquad \mu < 1, \end{aligned}$$

holds. Because the vector $\mathbf{n}_\Gamma$ is orthogonal to the vector $\mathbf{w}$,

- $\|T\mathbf{n}_\Gamma + \mathbf{w}\|_2^2 = T^2 + \|\mathbf{w}\|_2^2$

$\Rightarrow \ \|\mathbf{w}'\|_2 \le \mu \|\mathbf{w}\|_2.$

Remark

- It is proved the closed orbit is an asymptotically stable within W_α.
- The Poincaré map $\mathbf{w}_i = \varphi(T_{i-1}, \mathbf{w}_{i-1})$ is well defined since the number of crossing of the trajectory $\varphi(t, \mathbf{w}_i)$ $(\underline{T_w} < t < \overline{T_w})$ and Γ is one.
- In actual calculation, Assumption 1 and 2 are verified using numerical verification techniques, for example Lohner method, C^1-Lohner method, mean value form and so on.

Numerical examples

Rössler system

$$
\begin{cases}
\dfrac{dx}{dt} = -y - z \\
\dfrac{dy}{dt} = x + ay \\
\dfrac{dz}{dt} = b - z(c - x)
\end{cases}
$$

Rössler system

Initial values and parameter values

$$a = 1/5, b = 1/5, c = 11/5,$$
$$\tilde{\mathbf{w}} = (-2.918, 1.514, 0.08106)^T,$$
$$\tilde{T} = 11.4770.$$

$$\mathbf{n}_\Gamma = \begin{pmatrix} -0.4974 \\ -0.8652 \\ -0.06278 \end{pmatrix}.$$

Rössler system

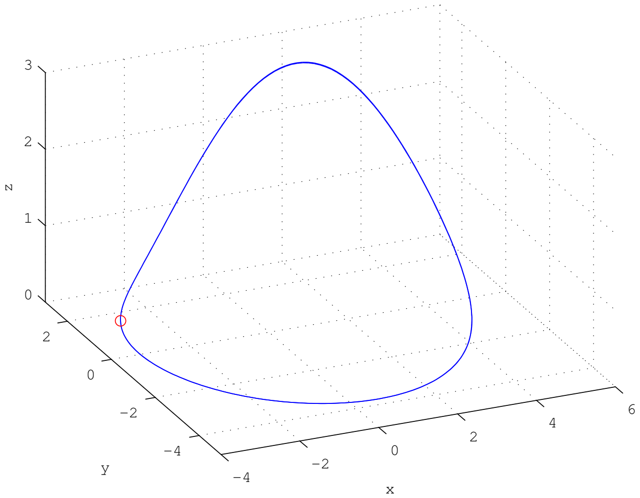

Figure: Approximate orbit

Verification for closed orbit existence

For $[Z] = ([T], [\mathbf{w}])^T$,

$$[Z] = \begin{pmatrix} [11.453898210, 11.4538982166] \\ [-0.001606575, -0.001606552] \\ [0.00094202, 0.000942042] \\ [-0.0002530964, -0.0002530926] \end{pmatrix}$$

$$N([Z]) = \begin{pmatrix} [11.453898211, 11.4538982161] \\ [-0.001606573, -0.001606554] \\ [0.00094203, 0.000942041] \\ [-0.0002530961, -0.0002530929] \end{pmatrix}$$

$\Rightarrow$ $N([Z]) \subset [Z]$

$\Rightarrow$ $\eta = 9.339 \times 10^{-9}$

Identification of basin

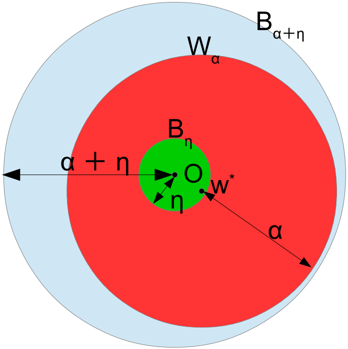

Figure: Illustration of W_α, B_α, $B_{\alpha+\eta}$

- B_η : the ball which includes a fixed point $\mathbf{w}^*$.
- W_α : a given ball which should be included by the basin of the closed orbit.
- $B_{\alpha+\eta}$: a ball on Γ where the assumptions are checked.

$$\eta = 9.339\times 10^{-9}.$$

$$\text{We } \textit{take } \alpha = 10^{-5} - \eta.$$

$B_{\alpha+\eta}$

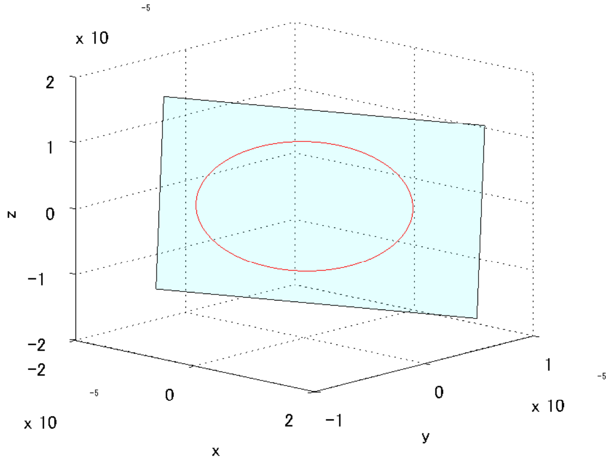

Figure: $B_{\alpha+\eta}$

Assumption 1

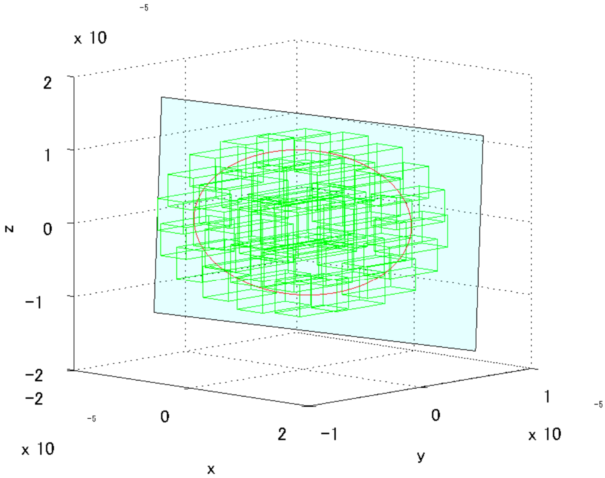

Figure: Boxes covering $B_{\alpha+\eta}$

Assumption 1

We obtained

$$\bigcup_{w \in B_{\alpha+\eta}} [T_w] = [11.47, 11.48].$$

.

Assumption 2

$$\|\mathbf{n}_\Gamma - \frac{\mathbf{f}(\varphi(T,\mathbf{w}))}{\|\mathbf{f}(\varphi(T,\mathbf{w}))\|_2}\|_2 = [0.001344, 0.001382],$$

$$\|\frac{\partial\varphi}{\partial\mathbf{w}}(T,\mathbf{w})P_\Gamma\|_2 = [0.9258, 0.9260],$$

$$\|\mathbf{n}_\Gamma - \frac{\mathbf{f}(\varphi(T,\mathbf{w}))}{\|\mathbf{f}(\varphi(T,\mathbf{w}))\|_2}\|_2 + \|\frac{\partial\varphi}{\partial\mathbf{w}}(T,\mathbf{w})P_\Gamma\|_2 = [0.9272, 0.9274].$$

Identification of basin

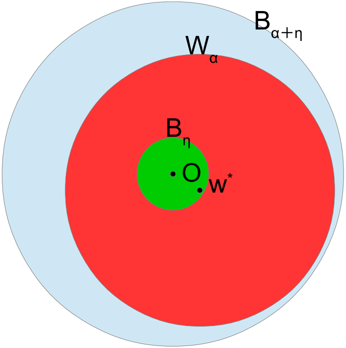

Figure: Illustration of W_α, B_α, $B_{\alpha+\eta}$

$$\eta = 9.339 \times 10^{-9}, \alpha = 10^{-5} - \eta$$

- There is a limit cycle in B_η, and its basin includes W_α.

Conclusion

Conclusion

- We developed a method to prove the closed orbit is asymptotically stable and to identify an area in the basin of a closed orbit.

Future works

- We will extend our methods for saddle type closed orbits to construct Lyapunov functions.

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