Reserve as recognizing functional for interval linear systems

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SCAN'2014, Würzburg, September 21–26, 2014

1. **Inclusion** $\boldsymbol{C}x \subseteq \boldsymbol{d}$
2. Reserve of $\boldsymbol{C}x \subseteq \boldsymbol{d}$
3. What can Reserve serve for?

Object under study

Our object under study is the interval linear inclusion

$$\boldsymbol{C}x \subseteq \boldsymbol{d}, \qquad \boldsymbol{C} \in \mathbb{KR}^{m\times n},\ \boldsymbol{d} \in \mathbb{K}\overline{\mathbb{R}}^{m\times n},\ x \in \mathbb{R}^n,$$

where $\mathbb{KR} = \{[\underline{\boldsymbol{v}}, \overline{\boldsymbol{v}}] \mid \underline{\boldsymbol{v}}, \overline{\boldsymbol{v}} \in \mathbb{R}\}$ is the set of Kaucher intervals, $\mathbb{K}\overline{\mathbb{R}} = \{[\underline{\boldsymbol{v}}, \overline{\boldsymbol{v}}] \mid \underline{\boldsymbol{v}}, \overline{\boldsymbol{v}} \in \overline{\mathbb{R}}\}$ is the set of Kaucher intervals over the extended real axis $\overline{\mathbb{R}} = \mathbb{R} \cup \{-\infty, \infty\}$.

The set of (formal) solutions is

$$\begin{aligned} \varXi &= \{x \in \mathbb{R}^n \mid \boldsymbol{C}x \subseteq \boldsymbol{d}\} \\ &= \Big\{x \in \mathbb{R}^n \mid \underline{\textstyle\sum \boldsymbol{C}_{ij}x_j} \geqslant \underline{\boldsymbol{d}}_i,\ \overline{\textstyle\sum \boldsymbol{C}_{ij}x_j} \leqslant \overline{\boldsymbol{d}}_i,\ i = 1, \dots, m\Big\}. \end{aligned}$$

Where the inclusion $\boldsymbol{C}x \subseteq \boldsymbol{d}$ is useful?

Quantifier solutions to interval problems

Usually, we consider intervals

- from the set of proper intervals
  $\mathbb{IR} = \{\boldsymbol{v} = [\underline{\boldsymbol{v}}, \overline{\boldsymbol{v}}] \mid \underline{\boldsymbol{v}}, \overline{\boldsymbol{v}} \in \mathbb{R},\ \underline{\boldsymbol{v}} \leqslant \overline{\boldsymbol{v}}\}$
- and in connection with a property (say, P) that can be fulfilled or not fulfilled for its point members.

Then the following different situations may occur:

1) either the property $P(v)$ holds for *all* members v from the given interval $\boldsymbol{v}$,
2) or the property $P(v)$ holds only for *some* members v from the interval $\boldsymbol{v}$,
   not necessarily all, or even for a single value.

Quantifier solutions to interval problems

Types of interval uncertainty

Formally, the above distinction can be expressed by logical quantifiers:

- In the first case, we write "$(\forall v \in \boldsymbol{v})\, P(v)$" and speak of *interval A-uncertainty*,
- In the second case, we write "$(\exists v \in \boldsymbol{v})\, P(v)$" and speak of *interval E-uncertainty*.

Quantifier solutions to interval problems

Interval system of relations

Let us consider the interval system of relations

$$F(\boldsymbol{a}, x)\ \sigma\ \boldsymbol{b}, \quad \sigma \in \{=, \leqslant, \geqslant\}^m,\ \boldsymbol{a} \in \mathbb{IR}^l,\ \boldsymbol{b} \in \mathbb{IR}^m,\ x \in \mathbb{R}^n,$$

with a selecting predicate

$$(Q_1 v_{\pi_1} \in \boldsymbol{v}_{\pi_1})(Q_2 v_{\pi_2} \in \boldsymbol{v}_{\pi_2}) \cdots (Q_{l+m} v_{\pi_{l+m}} \in \boldsymbol{v}_{\pi_{l+m}})\big(F(a, x)\, \sigma\, b\big)$$

where

$Q_1, Q_2, \ldots, Q_{l+m}$ — logical quantifiers "$\forall$" or "$\exists$",

$(v_1, v_2, \ldots, v_{l+m}) := (a_1, a_2, \ldots, a_l, b_1, b_2, \ldots, b_m) \in \mathbb{R}^{l+m}$
— aggregated parameter vector,

$(\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_{l+m}) := (\boldsymbol{a}_1, \boldsymbol{a}_2, \ldots, \boldsymbol{a}_l, \boldsymbol{b}_1, \boldsymbol{b}_2, \ldots, \boldsymbol{b}_m) \in \mathbb{IR}^{l+m}$
— aggregated interval vector of their possible values,

$(\pi_1, \pi_2, \ldots, \pi_{l+m})$ — a permutation of the integer numbers $1, 2, \ldots, l+m$.

Quantifier solutions to interval problems
Interval system of relations

A vector x will be referred to as quantifier solution to interval system of relations $F(\boldsymbol{a}, x)\ \sigma\ \boldsymbol{b}$ if the selecting predicate is true.

A quantifier solution for which, in the selecting predicate, all occurrences of the universal quantifier "$\forall$" precede those of the existential quantifier "$\exists$" will be referred to as AE-solution.

Particular cases of $\boldsymbol{C}x \subseteq \boldsymbol{d}$

The inclusion $\boldsymbol{C}x \subseteq \boldsymbol{d}$ is closely connected with the system of interval linear relations $\boldsymbol{A}x\,\boldsymbol{\sigma}\,\boldsymbol{b}$, where

$\sigma \in \{=, \leqslant, \geqslant\}^m$, $\boldsymbol{A} \in \mathbb{IR}^{m\times n}$, $\boldsymbol{b} \in \mathbb{IR}^m$, $x \in \mathbb{R}^n$,
$\mathcal{A} \in \{\forall, \exists\}^{m\times n}$ and $\beta \in \{\forall, \exists\}^m$ specify uncertainty types of the separate interval parameters $\boldsymbol{A}_{ij}$, $\boldsymbol{b}_i$ for all i and j.

We define the matrices $\boldsymbol{A}^\forall$, $\boldsymbol{A}^\exists$ and vectors $\boldsymbol{b}^\forall$, $\boldsymbol{b}^\exists$ as follows

$$\boldsymbol{A}^\forall_{ij} := \begin{cases} \boldsymbol{A}_{ij}, & \text{if } \mathcal{A}_{ij} = \forall, \\ 0, & \text{if } \mathcal{A}_{ij} = \exists, \end{cases} \qquad \boldsymbol{b}^\forall_{i} := \begin{cases} \boldsymbol{b}_{i}, & \text{if } \beta_{i} = \forall, \\ 0, & \text{if } \beta_{i} = \exists, \end{cases}$$

$$\boldsymbol{A}^\exists_{ij} := \begin{cases} \boldsymbol{A}_{ij}, & \text{if } \mathcal{A}_{ij} = \exists, \\ 0, & \text{if } \mathcal{A}_{ij} = \forall, \end{cases} \qquad \boldsymbol{b}^\exists_{i} := \begin{cases} \boldsymbol{b}_{i}, & \text{if } \beta_{i} = \exists, \\ 0, & \text{if } \beta_{i} = \forall. \end{cases}$$

Particular cases of $Cx \subseteq d$

AE-solutions to $Ax = b$

$$(\forall A' \in \boldsymbol{A}^{\forall})\,(\forall b' \in \boldsymbol{b}^{\forall})\,(\exists A'' \in \boldsymbol{A}^{\exists})\,(\exists b'' \in \boldsymbol{b}^{\exists})\,\big((A' + A'')x = b' + b''\big)$$

$$\Longleftrightarrow \quad \big(\boldsymbol{A}^{\forall} + \text{dual}\,\boldsymbol{A}^{\exists}\big)x \subseteq \text{dual}\,\boldsymbol{b}^{\forall} + \boldsymbol{b}^{\exists}.$$

S.P. Shary, A new technique in systems analysis under interval uncertainty and ambiguity, *Reliable Computing*, 8 (2002), No. 5, pp. 321–418.
http://interval.ict.nsc.ru/shary/Papers/ANewTech.pdf

Particular cases of $\boldsymbol{C}x \subseteq \boldsymbol{d}$

Quantifier solutions to $\boldsymbol{A}x \leqslant \boldsymbol{b}$ or $\boldsymbol{A}x \geqslant \boldsymbol{b}$

$Q(\boldsymbol{A}, \boldsymbol{b}, \mathcal{A}, \beta)$ is a quantifier prefix made up of the quantifier prefixes that correspond to the separate interval parameters. The order of the quantifiers is arbitrary.

$$Q(\boldsymbol{A}, \boldsymbol{b}, \mathcal{A}, \beta)(\boldsymbol{A}x \geqslant \boldsymbol{b}) \iff \left(\boldsymbol{A}^{\forall} + \text{dual}\, \boldsymbol{A}^{\exists}\right)x \subseteq [\overline{\boldsymbol{b}}^{\forall} + \underline{\boldsymbol{b}}^{\exists}, \infty]),$$

$$Q(\boldsymbol{A}, \boldsymbol{b}, \mathcal{A}, \beta)(\boldsymbol{A}x \leqslant \boldsymbol{b}) \iff \left(\boldsymbol{A}^{\forall} + \text{dual}\, \boldsymbol{A}^{\exists}\right)x \subseteq [-\infty, \underline{\boldsymbol{b}}^{\forall} + \overline{\boldsymbol{b}}^{\exists}]).$$

I.A. Sharaya, Quantifier-free descriptions of interval-quantifier linear systems, *Trudy Instituta Matematiki i Mekhaniki UrO RAN*, 20 (2014), No. 2, pp. 311–323. (In Russian)
http://interval.ict.nsc.ru/sharaya/Papers/trIMM14.pdf

Particular cases of $Cx \subseteq d$
Some quantifier solutions to $Ax \, \sigma \, b$

$Q^{\sigma}(\boldsymbol{A}, \boldsymbol{b}, \mathcal{A}, \beta)$ is a quantifier prefix made up of the quantifier prefixes that correspond to the separate interval parameters. For the interval parameters from the relation with "$=$", all quantifiers "$\forall$" precede all quantifiers "$\exists$".

$$Q^{\sigma}(\boldsymbol{A}, \boldsymbol{b}, \mathcal{A}, \beta)(\boldsymbol{A}x \, \sigma \, \boldsymbol{b}) \iff$$
$$\left(\boldsymbol{A}^{\forall} + \operatorname{dual} \boldsymbol{A}^{\exists}\right)x \subseteq \operatorname{dual} \boldsymbol{b}^{\forall} + \boldsymbol{b}^{\exists} + \boldsymbol{w},$$

where $\boldsymbol{w}_i := \begin{cases} 0, & \text{if } \sigma_i \text{ is "$=$"}, \\ [0, \infty], & \text{if } \sigma_i \text{ is "$\geqslant$"}, \\ [-\infty, 0], & \text{if } \sigma_i \text{ is "$\leqslant$"}. \end{cases}$

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ibidem

Particular cases of $\boldsymbol{C}x \subseteq \boldsymbol{d}$

The inclusion $\boldsymbol{C}x \subseteq \boldsymbol{d}$ allows us to study all the particular cases

- simultaneously and in a uniform way,
- by interval methods.

1. Inclusion $Cx \subseteq d$
2. **Reserve of $Cx \subseteq d$**
3. What can Reserve serve for?

Definition

By Reserve of the inclusion $\boldsymbol{C}x \subseteq \boldsymbol{d}$, we call the maximal number $\text{Rsv} \in \overline{\mathbb{R}}$ such that

$$\boldsymbol{C}x + [-\text{Rsv}, \text{Rsv}]\, e \ \subseteq \ \boldsymbol{d}$$

for the m-vector $e = (1\ 1\ \ldots\ 1)^\top$.

If $\text{Rsv} < 0$ then $[-\text{Rsv}, \text{Rsv}]$ is improper interval.

Formulas for Rsv

From the above definition, we can deduce

$$\begin{aligned}\mathsf{Rsv} &= \min_i \min\left\{\underline{\boldsymbol{C}_{i:}x} - \underline{\boldsymbol{d}}_i,\ -\overline{\boldsymbol{C}_{i:}x} + \overline{\boldsymbol{d}}_i\right\}\\ &= \min_i \min\left\{\underline{\boldsymbol{C}}_{i:}x^+ - \overline{\boldsymbol{C}}_{i:}x^- - \underline{\boldsymbol{d}}_i,\ -\overline{\boldsymbol{C}}_{i:}x^+ + \underline{\boldsymbol{C}}_{i:}x^- + \overline{\boldsymbol{d}}_i\right\}\\ &= \min_i \min\left\{\sum_{j=1}^{n} \boldsymbol{C}_{ij}^{-\operatorname{sgn} x_j} x_j - \underline{\boldsymbol{d}}_i,\ -\sum_{j=1}^{n} \boldsymbol{C}_{ij}^{\operatorname{sgn} x_j} x_j + \overline{\boldsymbol{d}}_i\right\},\end{aligned}$$

where $\quad x^+, x^- \in \mathbb{R}^n_+,\ x^+ = \max\{0, x\},\ x^- = \max\{0, -x\},$

$$\boldsymbol{C}_{ij}^{-\operatorname{sgn} x_j} = \begin{cases}\underline{\boldsymbol{C}}_{ij}, & \text{if } x_j \geqslant 0,\\ \overline{\boldsymbol{C}}_{ij}, & \text{otherwise,}\end{cases} \qquad \boldsymbol{C}_{ij}^{\operatorname{sgn} x_j} = \begin{cases}\overline{\boldsymbol{C}}_{ij}, & \text{if } x_j \geqslant 0,\\ \underline{\boldsymbol{C}}_{ij}, & \text{otherwise.}\end{cases}$$

Reserve as functional of x

For fixed $\boldsymbol{C}$, $\boldsymbol{d}$, we can consider $\mathrm{Rsv}(x) : \mathbb{R}^n \to \mathbb{R}$.

Properties of the functional $\mathrm{Rsv}(x)$:

- is defined on the entire $\mathbb{R}^n$,
- Lipschitz continuous,
- piecewise-linear,
- concave in each orthant of $\mathbb{R}^n$.

Example of Rsv(x)

For the system of interval equations $\boldsymbol{A}x = \boldsymbol{b}$

with selecting predicate $(\exists A \in \boldsymbol{A})(\exists b \in \boldsymbol{b})(Ax = b)$,

Reserve of characteristic inclusion $(\text{dual}\,\boldsymbol{A})x \subseteq \boldsymbol{b}$

as functional of x coincides with functional

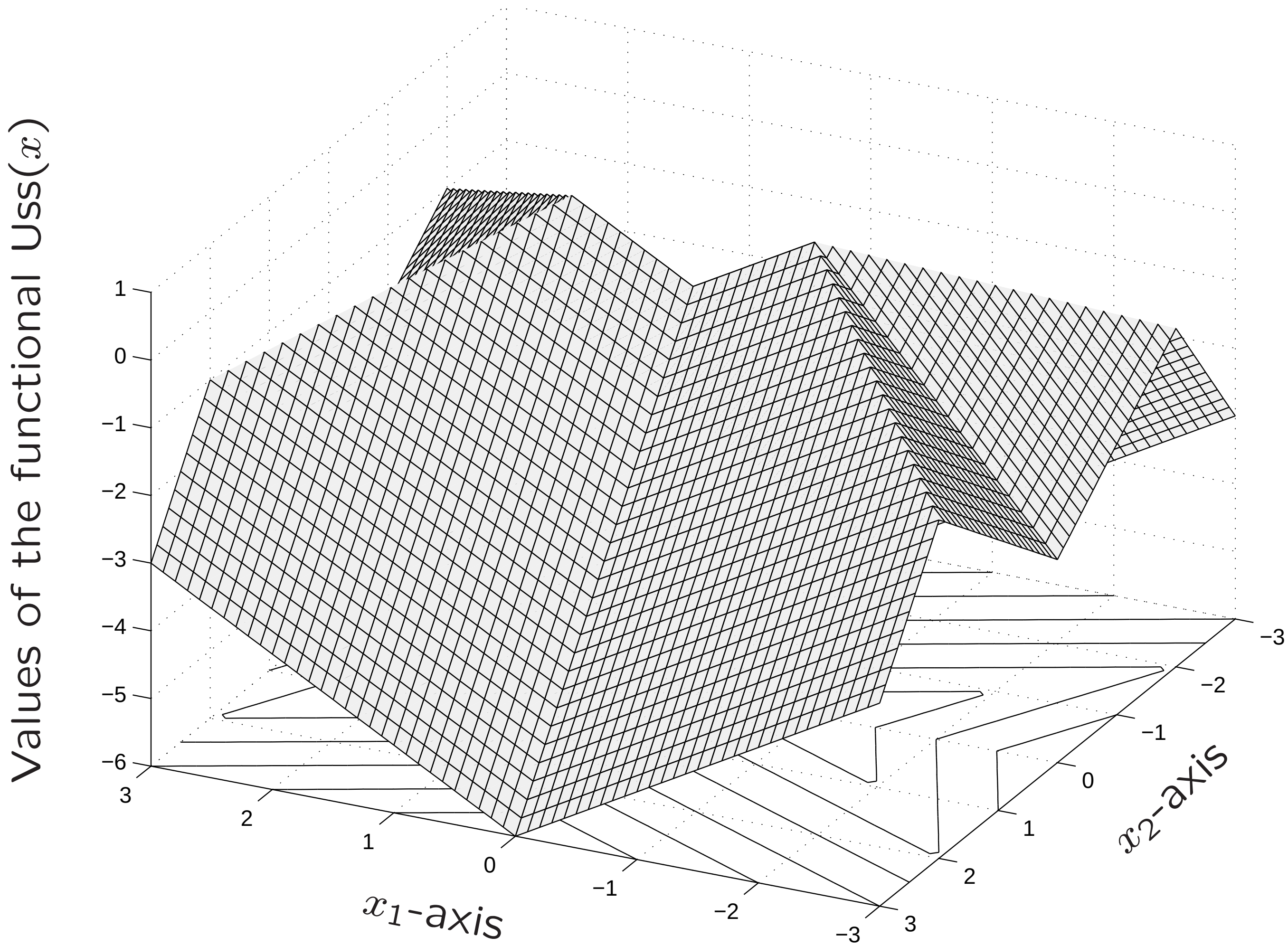
$$\text{Uss}(x) \;=\; \min_i \Big\{ \text{rad}\, \boldsymbol{b}_i + \text{rad}\, \boldsymbol{A}_{i:}\, |x| - \Big|\, \text{mid}\, \boldsymbol{b}_i - \text{mid}\, \boldsymbol{A}_{i:}\, x \,\Big| \Big\}.$$

(See S.P. Shary, Maximum consistency method for data fitting under interval uncertainty, *presentation at SCAN2014*, http://www.nsc.ru/interval/shary/Slides/Shary-SCAN2014.pdf)

Example of $\mathrm{Rsv}(x)$

Given the interval system $\begin{pmatrix} [2,4] & [-1,1] \\ [-1,1] & [2,4] \end{pmatrix} x = \begin{pmatrix} [-3,3] \\ 0 \end{pmatrix}$,

we have the functional



1. Inclusion $Cx \subseteq d$
2. Reserve of $Cx \subseteq d$
3. What can Reserve serve for?

Reserve as functional of x can recognize:

position of a point with respect to the solution set,

is the solution set Ξ empty or not,

is the interior of Ξ empty or not,

the 'best' points for the inclusion $\boldsymbol{C}x \subseteq \boldsymbol{d}$.

Position of a point with respect to the solution set

General case

From the definition of Reserve and properties of the functional Rsv(x), it is obvious that

$$\mathrm{Rsv}(\tilde{x}) \geqslant 0 \iff \tilde{x} \in \varXi,$$

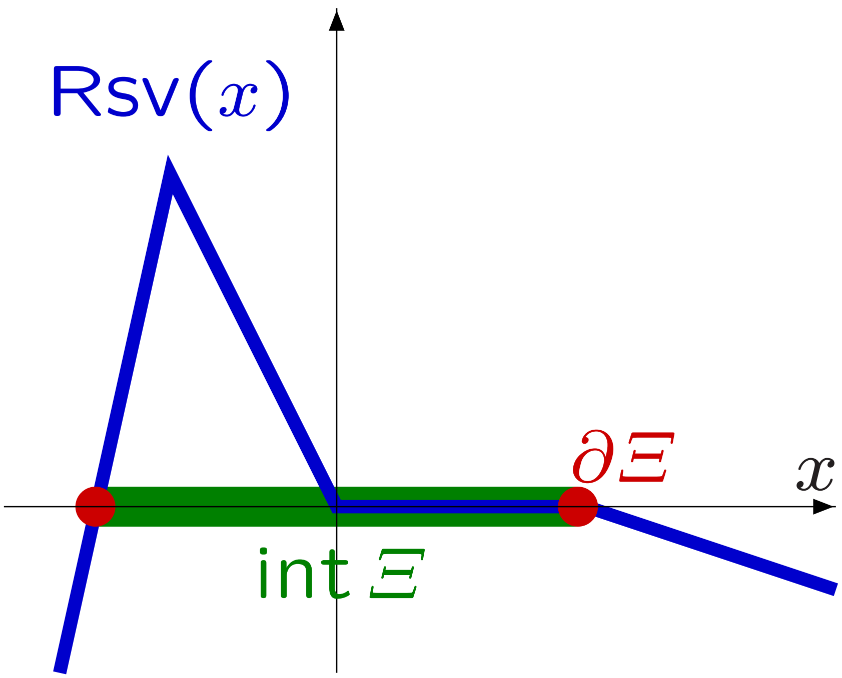
$$\mathrm{Rsv}(\tilde{x}) > 0 \implies \tilde{x} \in \mathrm{int}\,\varXi,$$

$$\mathrm{Rsv}(\tilde{x}) = 0 \impliedby \tilde{x} \in \partial\varXi,$$

where

$\mathrm{int}\,\varXi$ is topological interior of $\varXi$,

$\partial\varXi$ is boundary of $\varXi$.



Position of a point with respect to the solution set

Special conditions on C and d in the point

Notation: $L := \{i \mid \underline{\boldsymbol{C}_{i:}\tilde{x}} = \underline{\boldsymbol{d}}_i\}$, $R := \{i \mid \overline{\boldsymbol{C}_{i:}\tilde{x}} = \overline{\boldsymbol{d}}_i\}$,

$P := \{j \mid \tilde{x}_j > 0\}$,
$N := \{j \mid \tilde{x}_j < 0\}$,
$E := \{j \mid \tilde{x}_j = 0\}$.

Special conditions $\mathrm{SpeC}(\tilde{x})$:

$$\underline{\boldsymbol{C}}_{LP} = 0, \quad \overline{\boldsymbol{C}}_{LN} = 0, \qquad \boldsymbol{C}_{(L \cup R)E} \subseteq 0.$$
$$\overline{\boldsymbol{C}}_{RP} = 0, \quad \underline{\boldsymbol{C}}_{RN} = 0,$$

Proposition 1.

$$\tilde{x} \in \operatorname{int} \varXi \iff (\mathrm{Rsv}(\tilde{x}) > 0) \texttt{ OR } \big(\mathrm{Rsv}(\tilde{x}) = 0 \ \&\ \mathrm{SpeC}(\tilde{x})\big),$$

$$\tilde{x} \in \partial \varXi \iff \big(\mathrm{Rsv}(\tilde{x}) = 0 \ \&\ \neg \mathrm{SpeC}(\tilde{x})\big).$$

Position of a point with respect to the solution set

Special conditions on the point, C and d

Proposition 2. Let

1) at least one of the conditions be fulfilled
   - $\tilde{x}$ does not lie on the coordinate hyperplane,
   - the matrix $\boldsymbol{C}$ is proper,
2) and the augmented matrix $(\boldsymbol{C}, \boldsymbol{d})$ does not have rows with zero vertices.

Then

$$\tilde{x} \in \operatorname{int} \Xi \iff \operatorname{Rsv}(\tilde{x}) > 0,$$

$$\tilde{x} \in \partial \Xi \iff \operatorname{Rsv}(\tilde{x}) = 0.$$

Vertex of the vector $\boldsymbol{u} \in \mathbb{K}\overline{\mathbb{R}}^l$ is $u \in \overline{\mathbb{R}}^l$ such that $u_k \in \{\underline{\boldsymbol{u}}_k, \overline{\boldsymbol{u}}_k\}$.

If Rsv(x) is bounded from above,

then it reaches a finite maximum $\max_{x \in \mathbb{R}^n} \mathsf{Rsv}(x)$.

If Rsv(x) is unbounded from above,

we assume $\max_{x \in \mathbb{R}^n} \mathsf{Rsv}(x) = \infty$.

Notation (for brevity): $\max \mathsf{Rsv} := \max_{x \in \mathbb{R}^n} \mathsf{Rsv}(x)$.

Is the solution set empty or not?

(Solvability of the inclusion $\mathbf{C}x \subseteq \mathbf{d}$)

From $\quad \big(\, \mathrm{Rsv}(\tilde{x}) \geqslant 0 \iff \tilde{x} \in \varXi \,\big)$,

it follows that

$$\varXi \neq \varnothing \iff \max \mathrm{Rsv} \geqslant 0.$$

Is the interior of Ξ empty or not?

General case

From $\quad \big(\,\mathrm{Rsv}(\tilde{x}) > 0 \implies \tilde{x} \in \mathrm{int}\,\Xi\,\big),$

it follows that

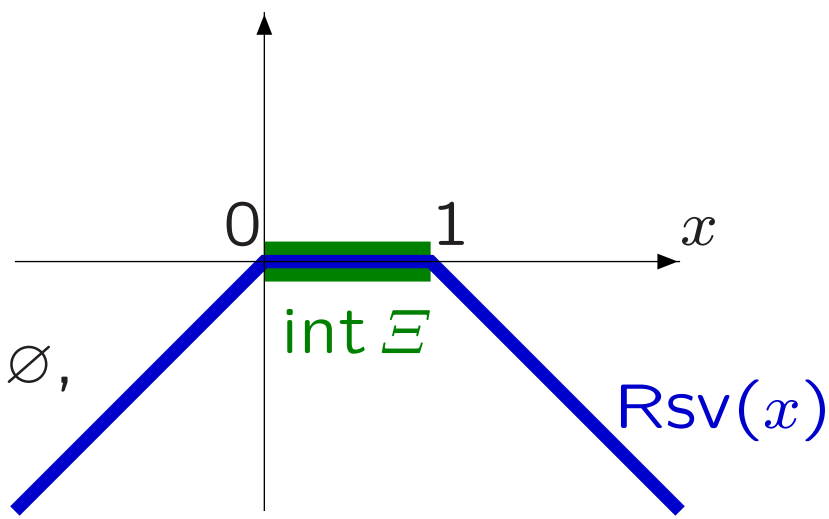
$$\max \mathrm{Rsv} > 0 \implies \mathrm{int}\,\Xi \neq \varnothing.$$

Example ($\not\Longleftarrow$).

For $\;[0,1]x \subseteq [0,1],$

we have $\;\mathrm{int}\,\Xi = \,]0,1[\, \neq \varnothing,$

but $\;\max \mathrm{Rsv} = 0.$



Is the interior of Ξ empty or not?

Special conditions on C and d

Proposition 3.

If the augmented matrix $(\boldsymbol{C}, \boldsymbol{d})$

does not have rows with zero vertices,

then

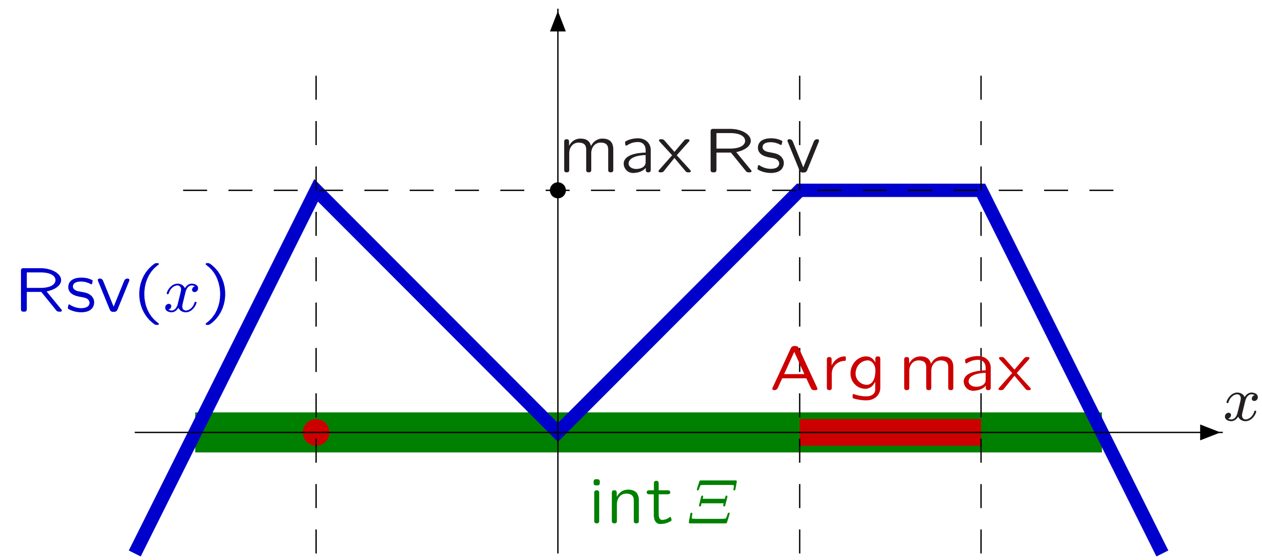
$$\operatorname{int} \Xi \neq \varnothing \iff \max \mathrm{Rsv} > 0.$$

The ‘best’ points for the inclusion $\boldsymbol{C}x \subseteq \boldsymbol{d}$

Notation: $\operatorname{Arg\,max} := \{y \in \mathbb{R}^n \mid \operatorname{Rsv}(y) = \max \operatorname{Rsv}\}$.

Case $\max \operatorname{Rsv} > 0$.

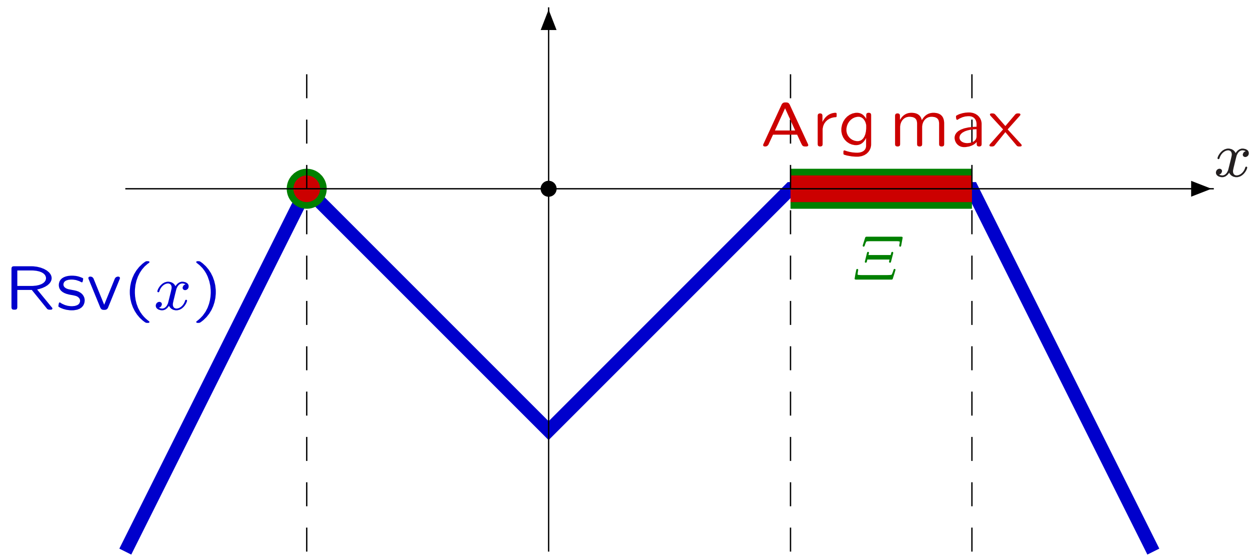
- Arg max consists of all such points for which $\boldsymbol{C}x \subseteq \boldsymbol{d}$ holds with maximum positive reserve.
- $\operatorname{Arg\,max} \subseteq \operatorname{int} \Xi$.



The 'best' points for the inclusion $\boldsymbol{C}x \subseteq \boldsymbol{d}$

Case $\max \mathrm{Rsv} = 0$.

- Arg max consists of all such points for which $\boldsymbol{C}x \subseteq \boldsymbol{d}$ holds with maximum reserve. But this reserve is 0.
- $\mathrm{Arg\,max} = \varXi$.

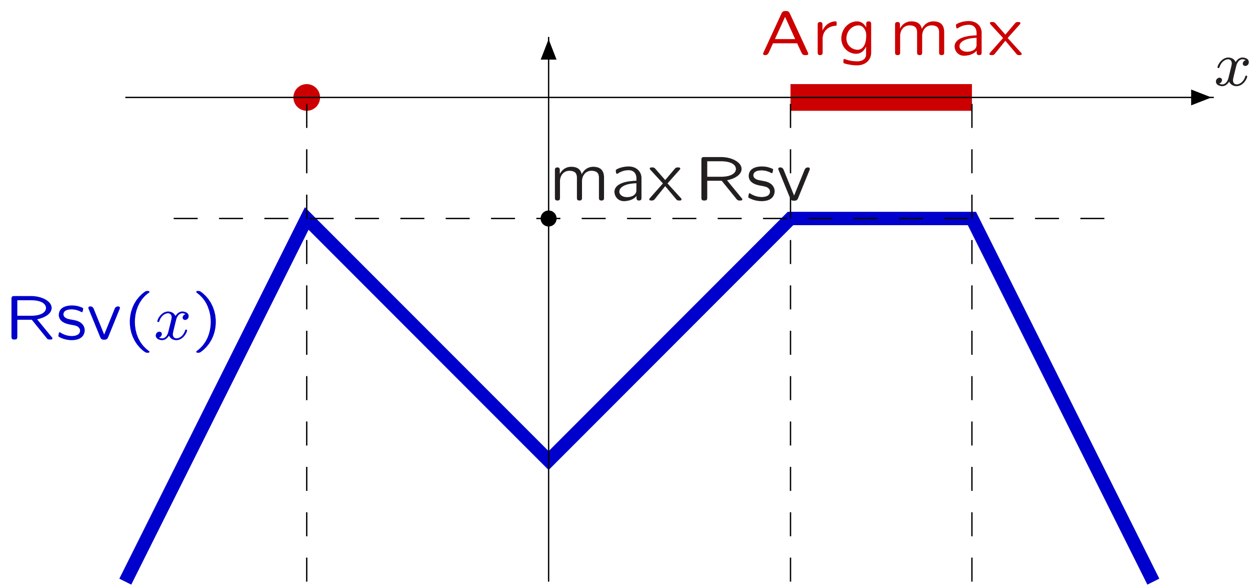


The ‘best’ points for the inclusion $\boldsymbol{C}x \subseteq \boldsymbol{d}$

Case $\max \mathrm{Rsv} < 0$.

- $\mathrm{Arg\,max}$ consists of all such points for which $\boldsymbol{C}x \subseteq \boldsymbol{d}$ is violated in the minimum amount.
- $\Xi = \varnothing$. $\mathrm{Arg\,max}$ is the solution set to $\boldsymbol{C}x \subseteq \boldsymbol{d} + e\,[\max \mathrm{Rsv}, -\max \mathrm{Rsv}]$.

One can use $\mathrm{Arg\,max}$ as set of ‘pseudosolutions’.



We wish you

positive reserve and

nonempty interior!