On a conjecture of Wright

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The Wright conjecture on a delay differential equation

Conjecture

The trajectories of the delay differential equation

$$y'(t) = -\alpha \left(e^{y(t-1)} - 1 \right) \tag{1}$$

converge to zero for $1.5 \leq \alpha \leq \pi/2$.

Citations

"My methods, at the cost of considerable elaboration, can be used to extend this result to $\alpha \leq \frac{37}{24}$ and, probably to $\alpha < 1.567...$ (compare with $\frac{\pi}{2} = 1.570796...$). But the work becomes so heavy for the last step that I have not completed it."

"... the authors using a combination of analytic and computational techniques, prove partially the long-standing Wright's conjecture ..."

The problem is closely related to the series of

$$y_1(x) = -1 - \sum_{n=1}^{\infty} c_n e^{n\alpha x}, \qquad y_2(x) = -1 - \sum_{n=1}^{\infty} (-1)^n c_n e^{n\alpha x}$$

with $c_1 = 1$, $(n-1)c_n = \sum_{m=1}^{n-1} c_m c_n m e^{-m\alpha}$.

$(1.5706 - 1.5)/(\pi/2 - 1.5) = 99.723\%$

Theoretical statements

Considering only those solutions of equation $y'(x) = -\alpha y(x-1)(1+y(x))$, $\alpha > 0$ which have values in $(-1, \infty)$, the transformation $x = \log(1+u)$ leads to the equation (1).

Proposition

Suppose $0 < \alpha \leq \pi/2$*. Then (1) has no homoclinic orbit to zero.*

Theorem

The zero solution of (1) is globally attracting if and only if (1) has no slowly oscillating periodic solution.

Theoretical statements 2

Corollary

If $0 < \alpha < \frac{\pi}{2}$ *and* $p^\alpha : \mathbb{R} \to \mathbb{R}$ *is a slowly oscillating periodic solution of equation (1) then*

$$\max_{t\in\mathbb{R}} p^\alpha(t) \geq \log\frac{\pi}{2\alpha} > 1 - \frac{2\alpha}{\pi}.$$

Theorem

If $\alpha \in [1.5, 1.5706]$, *then the zero solution of equation (1) is globally attractive.*

Theorem

If $\alpha \in [1.5, \pi/2]$ *and* $p^\alpha : \mathbb{R} \to \mathbb{R}$ *is a slowly oscillating periodic solution of (1), then* $\max_{t\in\mathbb{R}} |p^\alpha(t)| < 0.04$ *holds.*

Extension of the basic operations for intervals

Definition

Set theoretical definition:

$$A \circ B = \{a \circ b \mid a \in A \text{ and } b \in B\}, \; A, B \in \mathbb{I},$$

($\mathbb{I}$ is the set of compact intervals (i, j), where $i, j \in \mathbb{R}$, and $i \leq j$.)

Definition

Arithmetic definition:

$$[a, b] + [c, d] = [a + c, b + d],$$

$$[a, b] - [c, d] = [a - d, b - c],$$

$$[a, b] \cdot [c, d] = [\min(ac, ad, bc, bd), \max(ac, ad, bc, bd)],$$

$$[a, b]/[c, d] = [a, b] \cdot [1/d, 1/c] \;\text{ if }\; 0 \notin [c, d].$$

An example

The inclusion of the function

$$f(x) = x^2 - x$$

obtained for the interval $[0, 1]$ is $[-1, 1]$, while the range of it is here just $[-0.25, 0.0]$.

This eight times wider enclosure indicates that interval calculations should be useless. Still by using more sophisticated techniques the problem of the too loose enclosure can be overcome – at the cost of higher computational times.

Good bounds can easily be calculated also for the standard functions like sin, log, exp etc. too.

Implementation issues

In a floating point environment (most cases) the *outward rounding* is important to have a conservative inclusion that is a must in computer supported mathematical proofs.

Outward rounding means that the bounds of the calculated result intervals are rounded in such a way that all result points are within the given bounds.

In other words, the lower bound is always rounded toward $-\infty$, and the upper bound toward ∞. This can easily be realized applying the four rounding modes of the IEEE 754 standard (available on most programming languages and computers).

Implementation issues 2

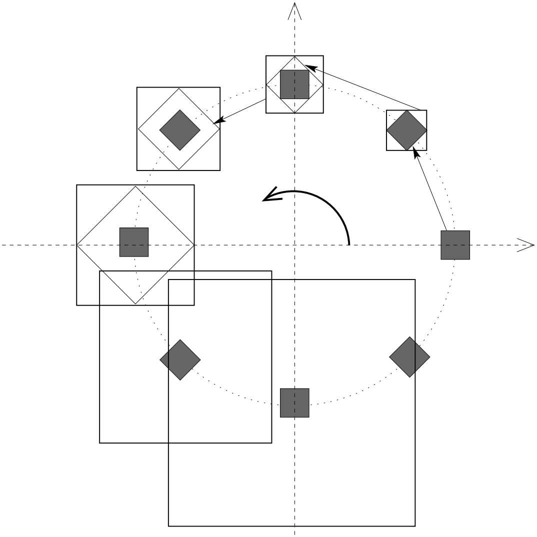
Several programming languages and packages support interval arithmetic based inclusion function generation: C-XSC, FORTRAN-XSC, PASCAL-XSC, and PROFIL/BIAS.

Interval packages are available in several numerical languages such as Maple, Mathematica, Matlab. The latter one, Intlab is especially easy to use.

By automatic differentiation the inclusions of first and second derivatives are easy to obtain, as well as multiple precision evaluations to achieve higher accuracy when necessary.

The interval function evaluations require 4 – 35 times more computation and time.

The wrapping effect



A verified differential equation solver

The VNODE program (Validated Solver for Initial Value Problems for Ordinary Differential Equations) by Ned Nedialkov (McMaster University) is a C++ programming language based package for computing rigorous bounds on the solution of an initial value problem for an ordinary differential equation.

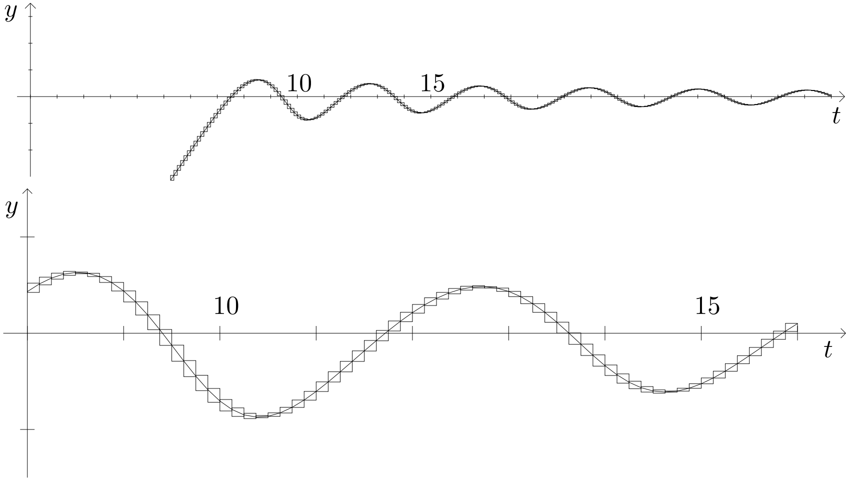
It is easy to implement, flexible, based on an Interval Hermite-Obreschkoff Method.

It provides no full remedy against the problems related to the wrapping effect.

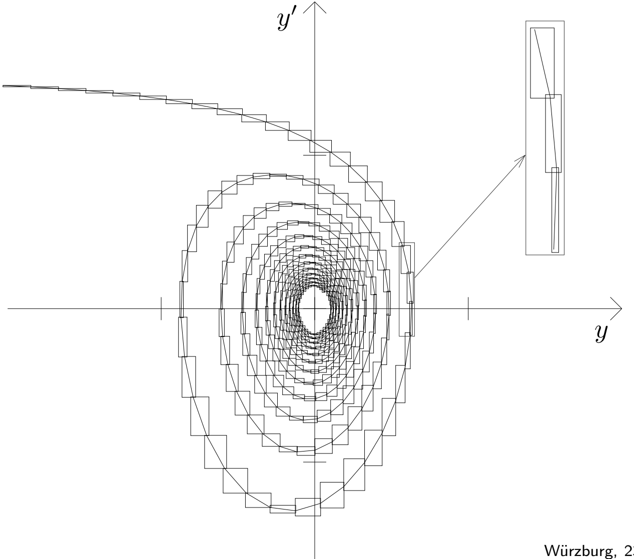
The program is available from

`http://www.cas.mcmaster.ca/~nedialk/Software/VNODE/VNODE.shtml`

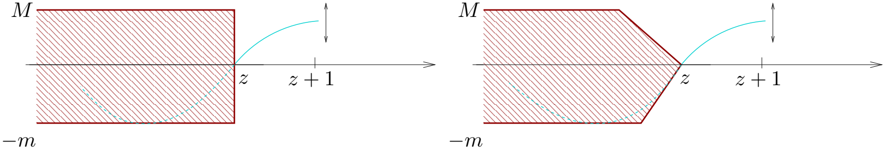
Inclusion of the trajectory for $\alpha = 1.5$



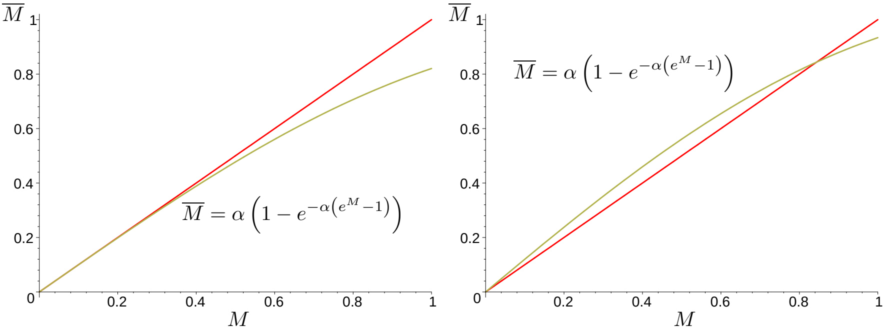
Inclusion of the trajectory for $\alpha = 1.5$ in the $y - y'$ plane



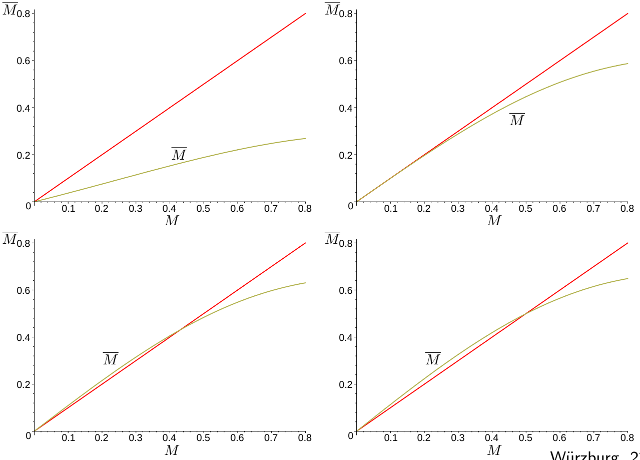
Regions where the solutions can be before z by using $-\alpha\left(e^{M}-1\right) \leq y'(t) \leq -\alpha\left(e^{-m}-1\right)$.



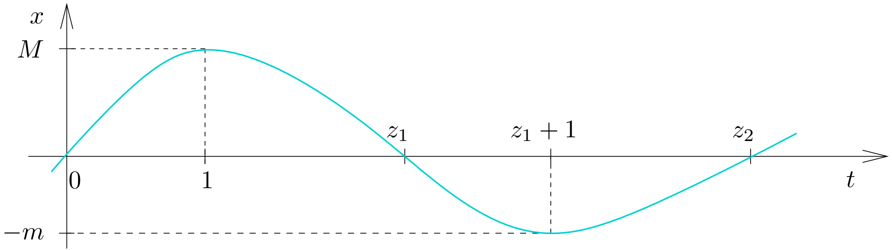
The simplest bounding function values compared to M for $\alpha = 1.0$ (left) and $\alpha = 1.1$ (right).



The second bounding function values compared to *M* for $\alpha = 1.1$ (top left), $\alpha = 1.5$ (top right), $\alpha = 1.55$ (bottom left), and $\alpha = \pi/2$ (bottom right).



The typical shape of a slowly oscillating periodic solution.



The bounding functions, denoted by dashed lines, are shifted to the part of the period where they bound p.

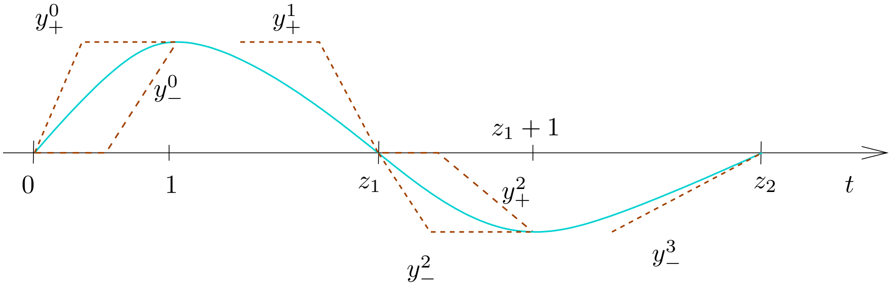
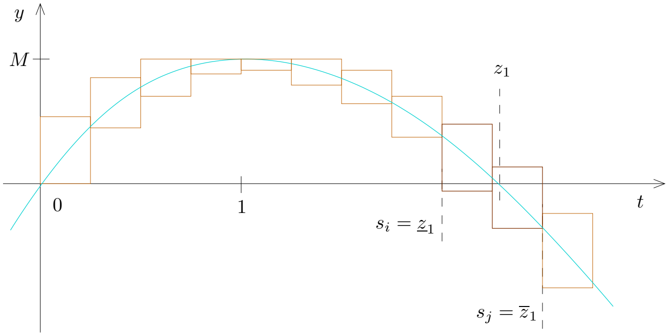
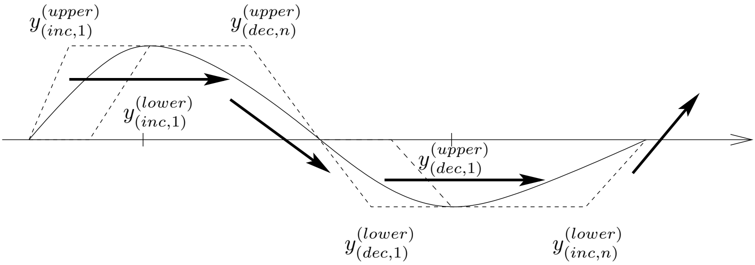


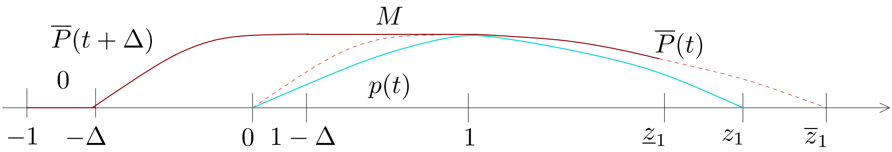
Illustration of the bounding procedure for the zero z_1 of p.



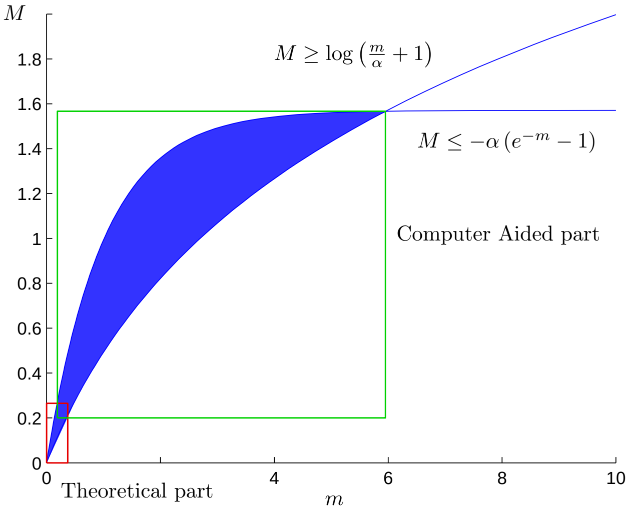
The bounding scheme



The definition of the upper bounding function based on the position of an unknown zero.



Regions to be proved by theory and by computer



The proven parts of the parameter space

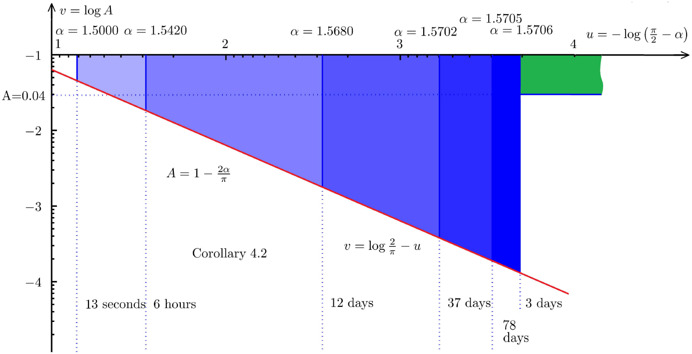


Illustration of the proved part of the conjecture, indicated by shades of blue (top left). The computation times are measured on a 4 core Sun Fire V490 workstation.

The proven parts of the parameter space 2

The convergence of the solutions to zero for $\alpha \in [1.5, 1.5420]$ was stated to be proven by E.M. Wright, but the proof was not given.

Wright thought that his technique can be successful until $\alpha = 1.567\ldots$

The green area in the top right corner demonstrates the virtual subproblem, in the case somebody could prove (e.g. by theoretical tools) that the slowly oscillating solutions cannot have an amplitude smaller than 0.04.

References

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The main paper can be downloaded from

`www.inf.u-szeged.hu/~csendes`