Error Estimations of Interpolations on Triangular Elements

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Contents

- The kth-order Lagrange interpolation on triangles
- The importance of error estimations
- The geometric conditions on triangles
- Kobayashi's formula and the circumradius condition
- The definition of surface area – Schwarz-Peano's example
- An extension
- Remarks

The kth-order Lagrange interpolation on triangles

k : a positive interger,

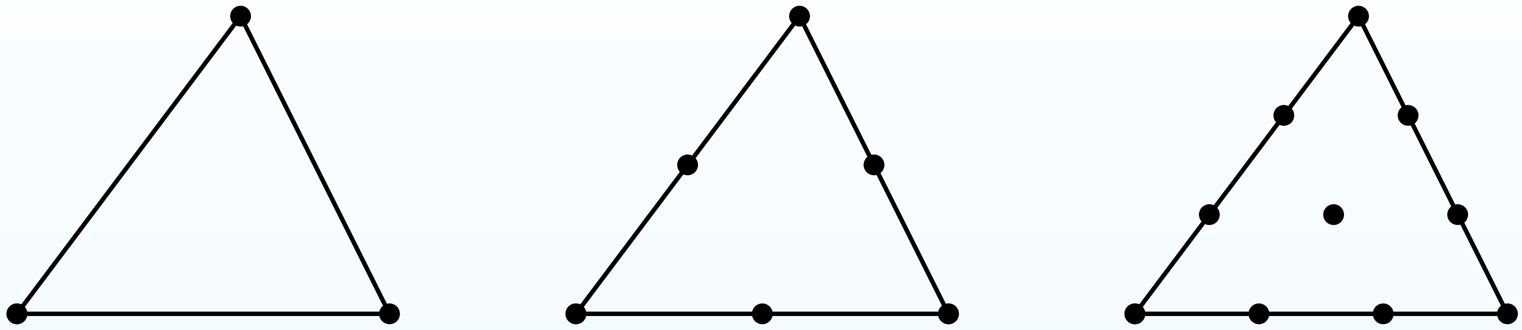
$\mathcal{P}_k$: the set of polynomials whose order are at most k,

$K \subset \mathbb{R}^2$: any triangle on $\mathbb{R}^2$,

$(\lambda_1, \lambda_2, \lambda_3)$: the barycentric coordinate on K,

a_i : integers,

$$\Sigma^k(K) := \left\{ \left(\frac{a_1}{k}, \frac{a_2}{k}, \frac{a_3}{k} \right) \in K \;\middle|\; 0 \le a_i \le k,\ a_1 + a_2 + a_3 = k \right\}.$$



K and $\Sigma^k(K)$, $k = 1$, $k = 2$, $k = 3$.

For $v \in C^0(\overline{K})$, define $\mathcal{I}_K^k v \in \mathcal{P}_k$ by

$$(\mathcal{I}_K^k v)(\mathbf{x}) = v(\mathbf{x}), \qquad \forall \mathbf{x} \in \Sigma^k(K).$$

We would like to estimate the error $\|v - \mathcal{I}_K^k v\|_{1,2,K}$.

The piecewise $\mathcal{P}_k$ finite element method

$\Omega \subset \mathbb{R}^2$: a bounded polygonal domain

τ : a proper triangulation of Ω

$$S_\tau := \left\{ v_h \in C^0(\overline{\Omega}) \cap H_0^1(\Omega) \;\middle|\; v|_K \in \mathcal{P}_k, \forall K \in \tau \right\}$$

Model problem Find $u \in H_0^1(\Omega)$ such that

$$-\Delta u = f \text{ for a given } f \in L^2(\Omega).$$

Weak form Find $u \in H_0^1(\Omega)$ such that

$$\int_\Omega \nabla u \cdot \nabla v \mathrm{d}x = \int_\Omega f v \mathrm{d}x \quad \text{for } \forall v \in H_0^1(\Omega).$$

$\mathcal{P}_k$ FEM Find $u_h \in S_\tau$ such that

$$\int_\Omega \nabla u_h \cdot \nabla v_h \mathrm{d}x = \int_\Omega f v_h \mathrm{d}x \quad \text{for } \forall v_h \in S_\tau.$$

Let $u \in H_0^1(\Omega) \cap H^2(\Omega)$ and $u_h \in S_\tau$ be the exact and finite element solutions, respectively. Then, by Céa's Lemma, we have

$$\begin{aligned} \|u - u_h\|_{1,2,\Omega} &\le C \inf_{v_h \in S_h} \|u - v_h\|_{1,2,\Omega} \\ &\le C\|u - \mathcal{I}_\tau^k u\|_{1,2,\Omega} \\ &= C \left(\sum_{K \in \tau} \|u - \mathcal{I}_K^k u\|_{1,2,K}^2 \right)^{1/2}, \end{aligned}$$

where C is a positive constant.

Therefore, estimating $\|u - \mathcal{I}_K^k u\|_{1,2,K}$ is very important in the error analysis of finite element methods.

Nakao's theory for numerical verifications of PDEs

For a given triangle $K \subset \mathbb{R}^2$, let $C(K)$ be the smallest constant such that

$$\|v - \mathcal{I}_K^k v\|_{1,2,K} \le C(K)|v|_{2,2,K} \quad \forall v \in H^2(K).$$

In Nakao's theory, it is very important to obtain a concrete value (or good upper bound) of $C(K)$ for a given triangle K.

Geometric Conditions on Triangles

It is known that we need to impose a geometric condition on triangles to obtain an error estimation.

- The minimum angle condition
- Shape-regularity
- The maximum angle condition
- Kobayashi's formula
- The circumradius condition

The minimum angle condition

Let h_K be the diameter (or the length of the longest edge) of K.

Theorem 1 *Let θ_0, $0 < \theta_0 < \pi/3$ be a constant. If any angle θ of K satisfies $\theta \geq \theta_0$, there exists a constant $C = C(\theta_0)$ independent of h_K such that, for $\forall h_K \leq h_0$,*

$$\|v - \mathcal{I}_K^1 v\|_{1,2,K} \leq C h_K |v|_{2,2,K}, \qquad \forall v \in H^2(K).$$

Zlámal, On the finite element method,
Numer. Math., **12** (1968) 394–409.

Shape-regularity

Let ρ_K be the diameter of the inscribed circle of K.

Theorem 2 *Let $\sigma > 0$ be a constant. If $h_K/\rho_K \le \sigma$, there exists a constant $C = C(\sigma)$ independent of h_K such that, for $\forall h_K \le h_0$,*

$$\|v - \mathcal{I}_K^1 v\|_{1,2,K} \le C h_K |v|_{2,2,K}, \qquad \forall v \in H^2(K).$$

Ciarlet, *The Finite Element Methods for Elliptic Problems,*
North Holland, 1978, reprint by SIAM 2008.
Brenner-Scott, *The Mathematical Theory of Finite Element Methods*,
3rd edition, Springer, 2008.

Note that the shape-regularity is equivalent to the minimum angle condition in $\mathbb{R}^2$

The maximum angle condition

Theorem 3 *Let* θ_1, $2\pi/3 < \theta_1 < \pi$ *be a constant. If any angle* θ *of* K *satisfies* $\theta \le \theta_1$, *there exists a constant* $C = C(\theta_1)$ *independent of* h_K *such that, for* $\forall h_K \le h_0$,

$$\|v - \mathcal{I}_K^1 v\|_{1,2,K} \le C h_K |v|_{2,2,K}, \qquad \forall v \in H^2(K).$$

Babuška-Aziz, On the angle condition in the finite element method,
SIAM J. Numer. Anal., **13** (1976) 214–226.

Kobayashi's formula

Let A, B, C be the lengths of edges of K and S be the area of K.

Theorem 4 *Let $C(K)$ be defined by*

$$C(K) := \sqrt{\frac{A^2B^2C^2}{16S^2} - \frac{A^2+B^2+C^2}{30} - \frac{S^2}{5}\left(\frac{1}{A^2}+\frac{1}{B^2}+\frac{1}{C^2}\right)},$$

then we have the following estimate:

$$|v - \mathcal{I}_K^1 v|_{1,2,K} \le C(K)|v|_{2,2,K}, \qquad \forall v \in H^2(K).$$

Kobayashi, On the interpolation constants over triangular elements (in Japanese), RIMS Kokyuroku, **1733** (2011), 58-77.

A corollary of Kobayashi's formula

Let R_K be the circumradius of K. Note that

$$R_K = \frac{ABC}{4S},$$

and therefore

$$C(K) := \sqrt{\frac{A^2B^2C^2}{16S^2} - \frac{A^2+B^2+C^2}{30} - \frac{S^2}{5}\left(\frac{1}{A^2}+\frac{1}{B^2}+\frac{1}{C^2}\right)} < R_K.$$

Corollary 5

$$|v - \mathcal{I}_K^1 v|_{1,2,K} \le R_K |v|_{2,2,K}, \qquad \forall v \in H^2(K).$$

The circumradius condition

Let R_K be the circumradius of K.

Theorem 6 *There exists a constant C_p independent of K such that, for $R_K \leq 1$,*

$$\|v - \mathcal{I}_K^1 v\|_{1,p,K} \leq C_p R_K |v|_{2,p,K}, \qquad \forall v \in W^{2,p}(K), \quad 1 \leq p \leq \infty.$$

Kobayashi-Tsuchiya, A Babuška-Aziz type proof of the circumradius condition, *Japan Journal of Industrial and Applied Mathematics*, **31** (2014) 193–210.

Rand, Delaunay refinement algorithms for numerical methods, Ph.D. dissertation, Carnegie Mellon University, 2009.

The circumradius condition (cont.)

Let $\{\tau_n\}_{n=1}^{\infty}$ be a sequence of triangulations of $\Omega \subset \mathbb{R}^2$ such that

$$\lim_{n\to\infty} R_n = 0, \qquad R_n := \max_{K\in\tau_n} R_K.$$

We say that $\{\tau_n\}_{n=1}^{\infty}$ satisfies the **circumradius condition** (of order 1).

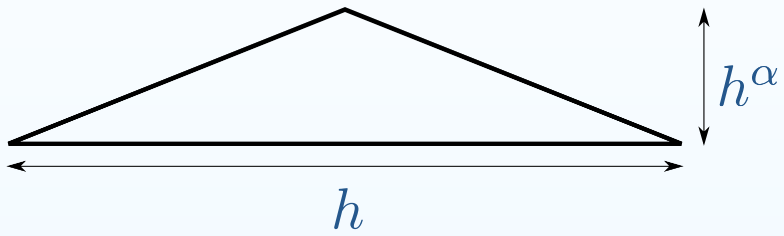
Let $u \in H_0^1(\Omega)$ and $u_n \in S_{\tau_n}$ be the exact and piecewise $\mathcal{P}_1$ finite element solutions of Poisson's equation $-\Delta u = f \in L^2(\Omega)$.

Corollary 7 *If $\{\tau_n\}$ satisfies the circumradius condition and $u \in H^2(\Omega)$, then we have*

$$\|u - u_n\|_{1,2,\Omega} \le C R_n |u|_{2,2,\Omega}.$$

The circumradius R_K and the maximum angle θ_K

Let $0 < h < 1$ and $\alpha > 1$. Consider the following isosceles triangle:



Let θ_K be the maximum angle of K. Note that $R_K = h^\alpha/2 + h^{2-\alpha}/8$.

Hence, if $1 < \alpha < 2$, we see

$$\lim_{h\to 0} R_K = 0 \quad \text{and} \quad \lim_{h\to 0} \theta_K = \pi.$$

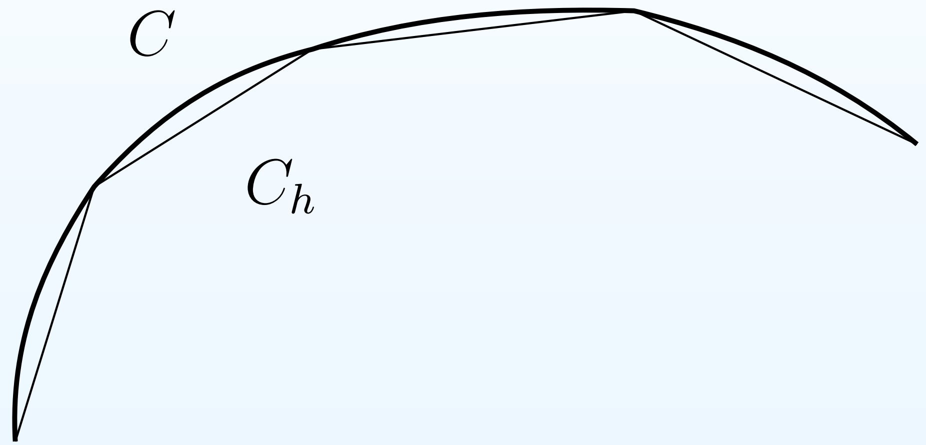
Therefore, *the circumradius condition is more general than the maximum angle condition*.

Hannukainen-Korotov-Křížek, The maximum angle condition is not necessary for convergence of the finite element method, Numer. Math., 120 (2011) 79–88.

The area of surfaces – The Schwarz-Peano paradox (1880)

Let C be a curve in $\mathbb{R}^n$, $n \geq 2$. The length $L(C)$ of C is defined by

$$L(C) := \sup L(C_h), \quad C_h \text{ is an inscribed polygonal curve.}$$

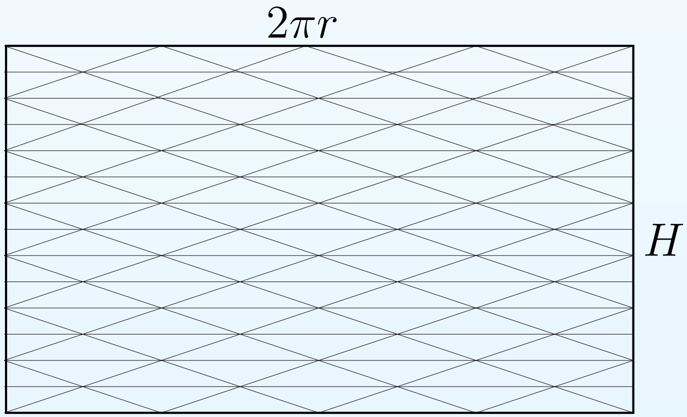


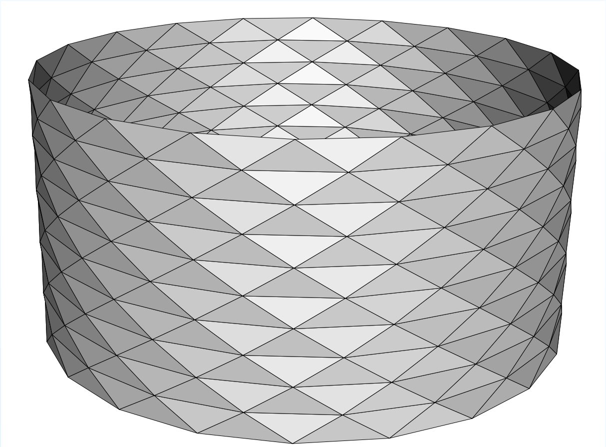
For a surface S, one might think we could define the area $A(S)$ by

$$A(S) := \sup A(S_h), \quad S_h \text{ is an inscribed polygonal (triangular) surface.}$$

In 1880, Schwarz and Peano independently showed that this definition does not work.

Consider the cylinder with radius r and height H. We triangulate the side of the cylinder as shown in the picture.





Let A_E be the sum of the area of all triangles. Then, we have

$$\begin{aligned} A_E &= 2mnr \sin\frac{\pi}{n} \sqrt{\left(\frac{H}{m}\right)^2 + r^2 \left(1 - \cos\frac{\pi}{n}\right)^2} \\ &= 2\pi r \frac{\sin\frac{\pi}{n}}{\frac{\pi}{n}} \sqrt{H^2 + \left(\frac{m}{n^2}\right)^2 \frac{\pi^4 r^2}{4} \left(\frac{\sin\frac{\pi}{2n}}{\frac{\pi}{2n}}\right)^4}. \end{aligned}$$

Therefore, when $n \to \infty$, $m \to \infty$, the value $\lim_{m,n\to\infty} A_E$ depends on the value $\lim_{m,n\to\infty}(m/n^2)$. In particular,

$$\lim_{m,n\to\infty} A_E = 2\pi r H \iff \lim_{m,n\to\infty} \frac{m}{n^2} = 0.$$

In the Schwarz-Peano example, the circumradius R of triangles is

$$R = \frac{\pi^2 r^2 \frac{m}{n^2} \left(\frac{\sin \frac{\pi}{2n}}{\frac{\pi}{2n}} \right)^2 + \frac{H^2}{m}}{2\sqrt{\frac{\pi^4 r^2}{4} \left(\frac{m}{n^2} \right)^2 \left(\frac{\sin \frac{\pi}{2n}}{\frac{\pi}{2n}} \right)^4 + H^2}}.$$

Therefore, when $m, n \to \infty$, we have

$$\lim_{m,n\to\infty} A_E = 2\pi r H \Longleftrightarrow \lim_{m,n\to\infty} \frac{m}{n^2} = 0 \Longleftrightarrow \lim_{m,n\to\infty} R = 0.$$

Kobayashi-Tsuchiya; On the circumradius condition for piecewise linear triangular elements, *submitted*, arXiv:1308.2113.

Extension to higher-order Lagrange interpolations

Let $\mathcal{T}_p^k(K)$ and $B_p^{m,k}(K)$ be defined by

$$\mathcal{T}_p^k(K) := \left\{ v \in W^{k+1,p}(K) \;\middle|\; v(\mathbf{x}) = 0, \ \forall \mathbf{x} \in \Sigma^k(K) \right\},$$

$$B_p^{m,k}(K) := \sup_{v \in \mathcal{T}_p^k(K)} \frac{|v|_{m,p,K}}{|v|_{k+1,p,K}}.$$

From the definitions, we have

$$v - \mathcal{I}_K^k v \in \mathcal{T}_p^k(K), \quad \forall v \in W^{k+1,p}(K),$$
$$|v - \mathcal{I}_K^k v|_{m,p,K} \le B_p^{m,k}(K) |v|_{k+1,p,K}.$$

Note that

$$B_p^{m,k}(K) = \inf \left\{ C; |v - \mathcal{I}_K^k v|_{m,p,K} \le C |v|_{k+1,p,K}, \ \forall v \in W^{k+1,p}(K) \right\}.$$

An extension to high-order Lagrange interpolations

Theorem 8 (Kobayashi-Tsuchiya) *Let $K \subset \mathbb{R}^2$ be an arbitrary triangle. Let R_K be the circumradius of K and $h_K := \mathrm{diam} K$. For any positive integer k and p, $1 \le p \le \infty$, there exists a constant $C_{k,p}$ independent of K such that, for $m = 0, 1, \cdots, k$ and $\forall v \in W^{k+1,p}(K)$,*

$$
\begin{aligned}
|v - \mathcal{I}_K^k v|_{m,p,K} &\le C_{k,p} R_K^m h_K^{k+1-2m} |v|_{k+1,p,K} \\
&= C_{k,p} \left(\frac{R_K}{h_K} \right)^m h_K^{k+1-m} |v|_{k+1,p,K}.
\end{aligned}
$$

Note that *no geometric condition is imposed on K.*

Kobayashi-Tsuchiya, Error estimates for Lagrange interpolations on triangles, *submitted*, arXiv:1408.2179

Remark 1

In many textbooks of the finite element methods, a triangulation τ of a domain $\Omega \subset \mathbb{R}^2$ is called shape-regular if

$$\frac{h_K}{\rho_K} \le C, \quad \forall K \in \tau.$$

It seems, however, that the ratio $\frac{R_K}{h_K}$ is more important than $\frac{h_K}{\rho_K}$.

Let $h_1 \le h_2 \le h_K$ be the length of edges of K and θ_K be the maximum angle of K. Observe

$$\frac{R_K}{h_K} = \frac{\frac{h_1 h_2 h_K}{4S}}{h_K} = \frac{h_1 h_2}{2h_1 h_2 \sin\theta_K} = \frac{1}{2\sin\theta_K},$$

$$\frac{R_K}{h_K} \le C \iff \theta_K \le \theta_0 < \pi$$: The Maximum Angle Condition!

Let $\{\tau_n\}_{n=1}^{\infty}$ be a sequence of triangulations of $\Omega \subset \mathbb{R}^2$ such that

$$\lim_{n\to\infty} R_n = 0, \qquad R_n := \max_{K\in\tau_n} R_K.$$

We say that $\{\tau_n\}_{n=1}^{\infty}$ satisfies the **circumradius condition** of order 1. Let $h_n := \max_{K\in\tau_n} h_K$.

Corollary 9 *Let $u \in H_0^1(\Omega)$ be an solution of the Poisson problem. Let u_n be the piecewise $\mathcal{P}_k$ finite element solution on τ_n. Supoose that $\{\tau_n\}$ satisfies the circumradius condition and $u \in H^{k+1}(\Omega)$, $k \geq 2$. Then we have*

$$\|u - u_n\|_{1,2,\Omega} \leq C R_n h_n^{k-1} |u|_{k+1,2,\Omega}.$$

If $\{\tau_n\}$ satisfies the maximum angle condition, we have

$$\|u - u_n\|_{1,2,\Omega} \leq C h_n^{k} |u|_{k+1,2,\Omega}.$$

Remark 2

Note that even if $\{\tau_n\}$ does not satisfy the circumradius condition of order 1, smaller h_K can overwhelm R_K.

Suppose that, for γ, $1 \leq \gamma < k$,

$$\lim_{n\to\infty} (Rh^{\gamma-1})_n = 0, \qquad (Rh^{\gamma-1})_n := \max_{K\in\tau_n} R_K h_K^{\gamma-1}.$$

We call the above condition the **circumradius condition** of order γ.

Theorem 10 *Let* $\{\tau_n\}_{n=1}^{\infty}$ *be an sequence of trianglulations of* $\Omega \subset \mathbb{R}^2$. *Let* $u \in H^{k+1}(\Omega)$ *be the exact solution, and* u_n *be the piecewise* $\mathcal{P}_k$ *finite element solution on* τ_n. *Suppose that* $\{\tau_n\}_{n=1}^{\infty}$ *satisfies the circumradius condition of order* γ, $1 \leq \gamma < k$. *Then we have*

$$\|u - u_n\|_{1,2,\Omega} \leq C(Rh^{\gamma-1})_n h_n^{k-\gamma} |u|_{k+1,2,\Omega}.$$

Related problems

- Develop a similar theory for Q_k-elements in 2-dim.
- Perform numerical experiments to see how finite element solutions of the (Navier-)Stokes equation behave under the circumradius condition.
- Find a good condition for tetrahedron or n-simplex, $n \geq 3$ similar to the circumradius condition.
- Error estimation of $\|u - u_h\|_{1,\infty,\Omega}$ under the circumradius condtion.
- ......

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