Error Estimations of Interpolations on Triangular Elements

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The $k$-th-order Lagrange interpolation on triangles

$k$: a positive integer,

$\mathcal{P}_k$: the set of polynomials whose order are at most $k$,

$K \subset \mathbb{R}^2$: any triangle on $\mathbb{R}^2$,

$(\lambda_1, \lambda_2, \lambda_3)$: the barycentric coordinate on $K$,

$a_i$: integers,

$$
\Sigma^k(K) := \left\{ \left( \frac{a_1}{k}, \frac{a_2}{k}, \frac{a_3}{k} \right) \in K \mid 0 \leq a_i \leq k, a_1 + a_2 + a_3 = k \right\}.
$$
For \( v \in C^0(\overline{K}) \), define \( \mathcal{I}_K^k v \in \mathcal{P}_k \) by

\[
(\mathcal{I}_K^k v)(x) = v(x), \quad \forall x \in \Sigma^k(K).
\]

We would like to estimate the error \( \| v - \mathcal{I}_K^k v \|_{1,2,K} \).
The piecewise $P_k$ finite element method

$\Omega \subset \mathbb{R}^2$: a bounded polygonal domain

$\tau$: a proper triangulation of $\Omega$

$S_\tau := \{ v_h \in C^0(\overline{\Omega}) \cap H^1_0(\Omega) \mid v|_K \in P_k, \forall K \in \tau \}$

**Model problem** Find $u \in H^1_0(\Omega)$ such that

$$-\Delta u = f \text{ for a given } f \in L^2(\Omega).$$

**Weak form** Find $u \in H^1_0(\Omega)$ such that

$$\int_\Omega \nabla u \cdot \nabla v \, dx = \int_\Omega fv \, dx \text{ for } \forall v \in H^1_0(\Omega).$$

$P_k$ FEM Find $u_h \in S_\tau$ such that

$$\int_\Omega \nabla u_h \cdot \nabla v_h \, dx = \int_\Omega fv_h \, dx \text{ for } \forall v_h \in S_\tau.$$
Let \( u \in H^1_0(\Omega) \cap H^2(\Omega) \) and \( u_h \in S_\tau \) be the exact and finite element solutions, respectively. Then, by Céa’s Lemma, we have

\[
\| u - u_h \|_{1,2,\Omega} \leq C \inf_{v_h \in S_h} \| u - v_h \|_{1,2,\Omega} \\
\leq C \| u - I^k \|_{1,2,\Omega} \\
= C \left( \sum_{K \in \tau} \| u - I^k_K u \|_{1,2,K}^2 \right)^{1/2},
\]

where \( C \) is a positive constant.
Therefore, estimating \( \| u - I^k_K u \|_{1,2,K} \) is very important in the error analysis of finite element methods.
For a given triangle $K \subset \mathbb{R}^2$, let $C(K)$ be the smallest constant such that

$$\|v - \mathcal{I}_K^k v\|_{1,2,K} \leq C(K)|v|_{2,2,K} \quad \forall v \in H^2(K).$$

In Nakao’s theory, it is very important to obtain a concrete value (or good upper bound) of $C(K)$ for a given triangle $K$. 
Geometric Conditions on Triangles

It is known that we need to impose a geometric condition on triangles to obtain an error estimation.

- The minimum angle condition
- Shape-regularity
- The maximum angle condition
- Kobayashi’s formula
- The circumradius condition
The minimum angle condition

Let $h_K$ be the diameter (or the length of the longest edge) of $K$.

**Theorem 1** Let $\theta_0$, $0 < \theta_0 < \pi/3$ be a constant. If any angle $\theta$ of $K$ satisfies $\theta \geq \theta_0$, there exists a constant $C = C(\theta_0)$ independent of $h_K$ such that, for $\forall h_K \leq h_0$,

$$\|v - I_K^1 v\|_{1,2,K} \leq C h_K |v|_{2,2,K}, \quad \forall v \in H^2(K).$$

Shape-regularity

Let $\rho_K$ be the diameter of the inscribed circle of $K$.

**Theorem 2** Let $\sigma > 0$ be a constant. If $h_K/\rho_K \leq \sigma$, there exists a constant $C = C(\sigma)$ independent of $h_K$ such that, for $\forall h_K \leq h_0$,

$$\|v - I^1_K v\|_{1,2,K} \leq C h_K |v|_{2,2,K}, \quad \forall v \in H^2(K).$$


Note that the shape-regularity is equivalent to the minimum angle condition in $\mathbb{R}^2$. 
The maximum angle condition

**Theorem 3** Let $\theta_1, 2\pi/3 < \theta_1 \leq \pi$ be a constant. If any angle $\theta$ of $K$ satisfies $\theta \leq \theta_1$, there exists a constant $C = C(\theta_1)$ independent of $h_K$ such that, for $\forall h_K \leq h_0$,

$$
\|v - \mathcal{I}_K^1 v\|_{1,2,K} \leq C h_K |v|_{2,2,K}, \quad \forall v \in H^2(K).
$$

Let $A, B, C$ be the lengths of edges of $K$ and $S$ be the area of $K$.

**Theorem 4** Let $C(K)$ be defined by

$$C(K) := \sqrt{\frac{A^2 B^2 C^2}{16S^2}} - \frac{A^2 + B^2 + C^2}{30} - \frac{S^2}{5} \left( \frac{1}{A^2} + \frac{1}{B^2} + \frac{1}{C^2} \right),$$

then we have the following estimate:

$$|v - \mathcal{I}_K^1 v|_{1,2,K} \leq C(K)|v|_{2,2,K}, \quad \forall v \in H^2(K).$$

A corollary of Kobayashi’s formula

Let $R_K$ be the circumradius of $K$. Note that

$$R_K = \frac{ABC}{4S},$$

and therefore

$$\sqrt{\frac{A^2 B^2 C^2}{16S^2}} - \frac{A^2 + B^2 + C^2}{30} - \frac{S^2}{5} \left( \frac{1}{A^2} + \frac{1}{B^2} + \frac{1}{C^2} \right) < R_K.$$

**Corollary 5**

$$|v - \mathcal{T}^1_K v|_{1,2,K} \leq R_K |v|_{2,2,K}, \quad \forall v \in H^2(K).$$
The circumradius condition

Let $R_K$ be the circumradius of $K$.

**Theorem 6** There exists a constant $C_p$ independent of $K$ such that, for $R_K \leq 1$,

$$\|v - \mathcal{I}_K^1 v\|_{1,p,K} \leq C_p R_K |v|_{2,p,K}, \quad \forall v \in W^{2,p}(K), \quad 1 \leq p \leq \infty.$$ 


The circumradius condition (cont.)

Let \( \{\tau_n\}_{n=1}^{\infty} \) be a sequence of triangulations of \( \Omega \subset \mathbb{R}^2 \) such that

\[
\lim_{n \to \infty} R_n = 0, \quad R_n := \max_{K \in \tau_n} R_K.
\]

We say that \( \{\tau_n\}_{n=1}^{\infty} \) satisfies the **circumradius condition** (of order 1).

Let \( u \in H^1_0(\Omega) \) and \( u_n \in S_{\tau_n} \) be the exact and piecewise \( \mathcal{P}_1 \) finite element solutions of Poisson’s equation \(-\Delta u = f \in L^2(\Omega)\).

**Corollary 7** If \( \{\tau_n\} \) satisfies the circumradius condition and \( u \in H^2(\Omega) \), then we have

\[
\|u - u_n\|_{1,2,\Omega} \leq CR_n \|u\|_{2,2,\Omega}.
\]
Let $0 < h < 1$ and $\alpha > 1$. Consider the following isosceles triangle:

Let $\theta_K$ be the maximum angle of $K$. Note that $R_K = h^\alpha/2 + h^{2-\alpha}/8$.

Hence, if $1 < \alpha < 2$, we see

\[
\lim_{h \to 0} R_K = 0 \quad \text{and} \quad \lim_{h \to 0} \theta_K = \pi.
\]

Therefore, the circumradius condition is more general than the maximum angle condition.

Let $C$ be a curve in $\mathbb{R}^n$, $n \geq 2$. The length $L(C)$ of $C$ is defined by

$$L(C) := \sup L(C_h), \quad C_h \text{ is an inscribed polygonal curve.}$$

For a surface $S$, one might think we could define the area $A(S)$ by

$$A(S) := \sup A(S_h), \quad S_h \text{ is an inscribed polygonal (triangular) surface.}$$
In 1880, Schwarz and Peano independently showed that this definition does not work.

Consider the cylinder with radius $r$ and height $H$. We triangulate the side of the cylinder as shown in the picture.
Let $A_E$ be the sum of the area of all triangles. Then, we have

$$A_E = 2mn r \sin \frac{\pi}{n} \sqrt{\left( \frac{H}{m} \right)^2 + r^2 \left( 1 - \cos \frac{\pi}{n} \right)^2}$$

$$= 2\pi r \frac{\sin \frac{\pi}{n}}{\frac{\pi}{n}} \sqrt{H^2 + \left( \frac{m}{n^2} \right)^2 \frac{\pi^4 r^2}{4} \left( \frac{\sin \frac{\pi}{2n}}\frac{\pi}{2n} \right)^4}.$$ 

Therefore, when $n \to \infty$, $m \to \infty$, the value $\lim_{m,n \to \infty} A_E$ depends on the value $\lim_{m,n \to \infty} (m/n^2)$. In particular,

$$\lim_{m,n \to \infty} A_E = 2\pi r H \iff \lim_{m,n \to \infty} \frac{m}{n^2} = 0.$$
In the Schwarz-Peano example, the circumradius $R$ of triangles is

$$R = \frac{\pi^2 r^2 m^2}{n^2} \left( \frac{\sin \frac{\pi}{2n}}{2n} \right)^2 + \frac{H^2}{m}.$$

Therefore, when $m, n \to \infty$, we have

$$\lim_{m,n \to \infty} A_E = 2\pi r H \iff \lim_{m,n \to \infty} \frac{m}{n^2} = 0 \iff \lim_{m,n \to \infty} R = 0.$$

Extension to higher-order Lagrange interpolations

Let $\mathcal{T}_p^k(K)$ and $B_p^{m,k}(K)$ be defined by

$$\mathcal{T}_p^k(K) := \left\{ v \in W^{k+1,p}(K) \mid v(x) = 0, \forall x \in \Sigma^k(K) \right\},$$

$$B_p^{m,k}(K) := \sup_{v \in \mathcal{T}_p^k(K)} \frac{|v|_{m,p,K}}{|v|_{k+1,p,K}}.$$

From the definitions, we have

$$v - I_K^k v \in \mathcal{T}_p^k(K), \quad \forall v \in W^{k+1,p}(K),$$

$$|v - I_K^k v|_{m,p,K} \leq B_p^{m,k}(K) |v|_{k+1,p,K}.$$

Note that

$$B_p^{m,k}(K) = \inf \left\{ C ; |v - I_K^k v|_{m,p,K} \leq C |v|_{k+1,p,K}, \forall v \in W^{k+1,p}(K) \right\}.$$
Theorem 8 (Kobayashi-Tsuchiya) Let $K \subset \mathbb{R}^2$ be an arbitrary triangle. Let $R_K$ be the circumradius of $K$ and $h_K := \text{diam}K$. For any positive integer $k$ and $p$, $1 \leq p \leq \infty$, there exists a constant $C_{k,p}$ independent of $K$ such that, for $m = 0, 1, \cdots, k$ and $\forall v \in W^{k+1,p}(K)$,

$$|v - \mathcal{I}_K^k v|_{m,p,K} \leq C_{k,p} R_K^m h_K^{k+1-2m} |v|_{k+1,p,K}$$

$$= C_{k,p} \left( \frac{R_K}{h_K} \right)^m h_K^{k+1-m} |v|_{k+1,p,K}.$$ 

Note that no geometric condition is imposed on $K$.

Remark 1

In many textbooks of the finite element methods, a triangulation \( \tau \) of a domain \( \Omega \subset \mathbb{R}^2 \) is called shape-regular if

\[
\frac{h_K}{\rho_K} \leq C, \quad \forall K \in \tau.
\]

It seems, however, that the ratio \( \frac{R_K}{h_K} \) is more important than \( \frac{h_K}{\rho_K} \).

Let \( h_1 \leq h_2 \leq h_K \) be the length of edges of \( K \) and \( \theta_K \) be the maximum angle of \( K \). Observe

\[
\frac{R_K}{h_K} = \frac{h_1 h_2 h_K}{4S} = \frac{h_1 h_2}{2h_1 h_2 \sin \theta_K} = \frac{1}{2 \sin \theta_K},
\]

\[
\frac{R_K}{h_K} \leq C \iff \theta_K \leq \theta_0 < \pi : \text{The Maximum Angle Condition!}
\]
Let \( \{ \tau_n \}_{n=1}^{\infty} \) be a sequence of triangulations of \( \Omega \subset \mathbb{R}^2 \) such that

\[
\lim_{n \to \infty} R_n = 0, \quad R_n := \max_{K \in \tau_n} R_K.
\]

We say that \( \{ \tau_n \}_{n=1}^{\infty} \) satisfies the **circumradius condition** of order 1. Let \( h_n := \max_{K \in \tau_n} h_K \).

**Corollary 9**  Let \( u \in H^1_0(\Omega) \) be an solution of the Poisson problem. Let \( u_n \) be the piecewise \( P_k \) finite element solution on \( \tau_n \). Suppose that \( \{ \tau_n \} \) satisfies the circumradius condition and \( u \in H^{k+1}(\Omega), \ k \geq 2 \). Then we have

\[
\| u - u_n \|_{1,2,\Omega} \leq CR_n h_n^{k-1} |u|_{k+1,2,\Omega}.
\]

If \( \{ \tau_n \} \) satisfies the maximum angle condition, we have

\[
\| u - u_n \|_{1,2,\Omega} \leq Ch_n^k |u|_{k+1,2,\Omega}.
\]
Remark 2

Note that even if \( \{\tau_n\} \) does not satisfy the circumradius condition of order 1, smaller \( h_K \) can overwhelm \( R_K \).

Suppose that, for \( \gamma, 1 \leq \gamma < k \),

\[
\lim_{n \to \infty} (Rh^{\gamma-1})_n = 0, \quad (Rh^{\gamma-1})_n := \max_{K \in \tau_n} R_K h_K^{\gamma-1}.
\]

We call the above condition the **circumradius condition** of order \( \gamma \).

**Theorem 10** Let \( \{\tau_n\}_{n=1}^{\infty} \) be a sequence of triangulations of \( \Omega \subset \mathbb{R}^2 \). Let \( u \in H^{k+1}(\Omega) \) be the exact solution, and \( u_n \) be the piecewise \( P_k \) finite element solution on \( \tau_n \). Suppose that \( \{\tau_n\}_{n=1}^{\infty} \) satisfies the circumradius condition of order \( \gamma, 1 \leq \gamma < k \). Then we have

\[
\|u - u_n\|_{1,2,\Omega} \leq C(Rh^{\gamma-1})_n h_n^{k-\gamma} |u|_{k+1,2,\Omega}.
\]
Related problems

- Develop a similar theory for $Q_k$-elements in 2-dim.
- Perform numerical experiments to see how finite element solutions of the (Navier-)Stokes equation behave under the circumradius condition.
- Find a good condition for tetrahedron or $n$-simplex, $n \geq 3$ similar to the circumradius condition.
- Error estimation of $\|u - u_h\|_{1,\infty,\Omega}$ under the circumradius condition.
- ......
References