Numerical Verification for Elliptic Boundary Value Problem with Nonconforming $\mathcal{P}_1$ Finite Elements

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Motive

Preceding studies: Nakao's Theory

- M. T. Nakao developed a method to verify the existence of solutions to an elliptic boundary value problem [7, Nakao '88].
- **Computer Assisted Analysis** for PDEs

Motive: to generalize Nakao's Method

- Nakao's method implicitly assumed the finite element space to be **conforming**.
- Interested in Nakao's Method with **nonconforming** $\mathcal{P}_1$ FEM.

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Motive

Why do we use nonconforming FEM?

- In 2D elasticity problem, convergence of conforming FEM is slow (**Rocking Effect**) [1, Babuška *et al.* '92].
- A device to avoid Rocking Effect: nonconforming FEM [3, Lee *et al.* '03], [5, Lovadina '05].
- In such cases, engineers use nonconforming FEM in structual analysis software tools.

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§.3 Nakao's Method with nonconforming FEs

- We generalize the algorithm for the nonconforming $\mathcal{P}_1$ FE.
- We obtain constructive inequalities for integrations on element boundaries which come from nonconforming FE.
- We show numerical results of our proposal method.

Section 1

Preliminaries

Subsections

Notations

- Let $\mathbf{IR} := \{ [a, b] \mid a \leq b \}$.
- Let $\Omega \subset \mathbf{R}^2$ be a bounded convex polygon.
- For simplicity, suppose $\Omega := (0, 1)^2$.

Notations

Notation. Inner products and norms on Hilbert spaces

$$(u,v)_{L^2(\Omega)} := \int_\Omega uv \,\mathrm{d}x \qquad (u,v \in L^2(\Omega))$$

$$(u,v)_{L^2(\Omega)} := \int_\Omega u\cdot v \,\mathrm{d}x \qquad (u,v \in L^2(\Omega)^2)$$

$$|u|_{H^1(\Omega)}^2 := \sum_{i=1}^{2} \|\partial_{x_i} u\|_{L^2(\Omega)}^2 \qquad (u \in H^1(\Omega))$$

$$|u|_{H^2(\Omega)}^2 := \sum_{i,j=1}^{2} \|\partial_{x_i}\partial_{x_j} u\|_{L^2(\Omega)}^2 \qquad (u \in H^2(\Omega))$$

$$\|u\|_{H^2(\Omega)}^2 := \|u\|_{L^2(\Omega)}^2 + |u|_{H^1(\Omega)}^2 + |u|_{H^2(\Omega)}^2 \qquad (u \in H^2(\Omega))$$

Notations

- Let $\{\mathcal{T}_h\}_h$ be regular triangulations.
- For simplicity, suppose $\mathcal{T}_{1/N}$ is the $\mathcal{P}_1$ triangulation shown in the figure.

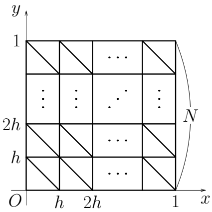

Figure: Triangulation $\mathcal{T}_h$

FEM

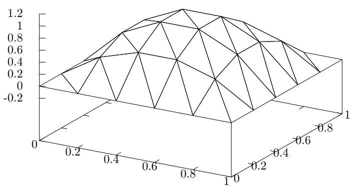

Figure: numerical solution by conforming $\mathcal{P}_1$ FEM

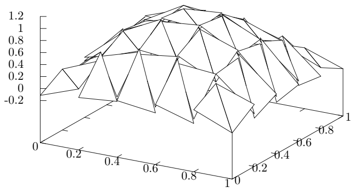

Figure: numerical solution by nonconforming $\mathcal{P}_1$ FEM

Nonconforming $\mathcal{P}_1$ FE

- A **node** is a midpoint of each edge of each triangle $T \in \mathcal{T}_h$.
- Let φ_i be a piecewise linear function s.t.
  - $\varphi_i = 1$ on the i-th node,
  - $\varphi_i = 0$ on other nodes,
  - continuous in each triangle $T \in \mathcal{T}_h$,
  - continuous at each node.

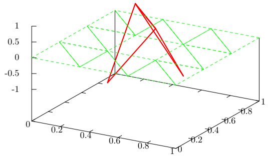

Figure: Nonconforming $\mathcal{P}_1$ basis function φ_i

Nonconforming $\mathcal{P}_1$ FE

Definition. Nonconforming $\mathcal{P}_1$ FE space

Let N_{h0} be the number of interior nodes in Ω.

$$X_{h0} := \mathrm{span}\left\{\varphi_1, \ldots, \varphi_{N_{h0}}\right\}.$$

Notation. Inner product on the FE space

For each $u, v \in X_{h0} + H_0^1(\Omega)$,

$$(u, v)_h := \sum_{T \in \mathcal{T}_h} (\nabla u, \nabla v)_{L^2(T)} .$$

$\left(X_{h0} + H_0^1(\Omega), (\,\cdot\,,\,\cdot\,)_h\right)$ is a Hilbert space [2].

Section 2

Nakao's method

Subsections

Configuration

Let $f\colon H_0^1(\Omega) \to L^2(\Omega)$ be a confinuous map.

2D elliptic BVP

Find $u \in H^2(\Omega)$ s.t.

$$\begin{cases} -\triangle u = f(u) & \text{in } \Omega\,, \\ u = 0 & \text{on } \partial\Omega\,. \end{cases}$$

Equivalent fixpoint problem

Find $u \in H_0^1(\Omega)$ s.t.

$$u = -\triangle^{-1} f(u)\,.$$

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Idea

Let $Y_{h0} \subset H_0^1(\Omega)$ be a conforming FE space and $u_h \in Y_{h0}$ be a numerical solution obtained by FEM.

- We expect there exists an exact solution nearby u_h.
- We want to verify existence of an exact solution in a "candidate" set U. We use **Schauder's fixpoint theorem** to verify the existence.
- For applying the fixpoint theorem, we now formulate sufficient conditions which we can check rigorously in computers.

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- For applying the fixpoint theorem, we now formulate sufficient conditions which we can check rigorously in computers.

Definition. Candidate set

Let $U \subset H_0^1(\Omega)$ be a closed convex set s.t.:

$$\begin{aligned} U &= u_h + U_h + U_*\,, \\ U_h &= \sum_{i=1}^{N_{h0}} U_i \varphi_i \subset Y_{h0} \qquad\qquad (U_i \in \mathbf{IR}\,,\ \mathbf{U}_h = {}^t(U_i)_i)\,, \\ U_* &= \left\{ u_* \in Y_{h0}^\perp \;\middle|\; \|u\|_h \le \alpha \right\} \subset Y_{h0}^\perp \qquad\qquad (\alpha > 0)\,. \end{aligned}$$

We here call U a candidate set.

Let $P_h \colon H_0^1(\Omega) \to Y_{h0}$ be the orthogonal projection onto Y_{h0}.

Infinite dimensional part + finite dimensional part

$-\triangle^{-1} f(U) \subset U$ holds if,

$$-(I - P_h) \triangle^{-1} f(U) \subset U_* \,, \tag{A}$$

$$-P_h \triangle^{-1} f(U) \subset u_h + U_h \,. \tag{B}$$

Sufficient condition for infinite dimensional part

Theorem. Sufficient condition for infinite dimensional part

Let Π_h^c be the conforming $\mathcal{P}_1$ interpolator from $H^2(\Omega) \cap H_0^1(\Omega)$ onto Y_{h0}. Then, (A) holds if,

$$\|I - \Pi_h^c\| \sup_{u \in U} \|f(u)\|_{L^2} \leq \alpha\,. \tag{1}$$

Remark.

- If we know an explicit upper bound $C(h)$ of $\|I - \Pi_h^c\|$ and one of $\sup\|f(u)\|$, we can check (1) rigorously in computers.
- $C(h) \to 0$ as $h \to 0$. Therefore, we expect (1) to hold if h small enough.

Sufficient condition for finite dimensional part

Theorem. Sufficient condition for finite dimensional part

Let $D_{ij} := (\nabla\varphi_j, \nabla\varphi_i)_{L^2}$. Suppose interval vectors $\mathbf{d} = {}^t(d_i)_i \in \mathbf{IR}^{N_{h0}}$ and $\mathbf{V}_h \in \mathbf{IR}^{N_{h0}}$ satisfy $D^{-1}\mathbf{d} \subset \mathbf{V}_h$ and

$$\{ (f(u), \varphi_i)_{L^2} - (u_h, \varphi_i)_h \mid u \in U \} \subset d_i \qquad (i = 1, \dots, N_{h0}). \tag{2}$$

Then, (B) holds if,

$$\mathbf{V}_h \subset \mathbf{U}_h .$$

Remark.

We can calculate $\mathbf{V}_h$ by Rump's algorithm [8, Rump '93].

Algorithm

- Calculate a next candidate set V by (1) and (2).
- $V \subset U \Longrightarrow$ O.K.
- If not, $U \leftarrow V$ and iterate.
- When iterating, it is good idea to use ε-inflation [9, Rump '98].

$u_h \leftarrow$ numerical solution by FEM
$\mathbf{U}_h \leftarrow$ some interval vector
$\alpha \leftarrow$ some positive value
for $i = 0$ to MAXIT **do**
    $\beta \leftarrow$ L.H.S. of (1)
    $\mathbf{d} \leftarrow$ L.H.S. of (2)
    $\mathbf{V}_h \leftarrow D^{-1}\mathbf{d}$ (Rump's algorithm)
    if $\mathbf{V}_h \subset \mathbf{U}_h \;\&\; \beta \le \alpha$ **then**
        Succeeded
    else
        $(\mathbf{U}_h, \alpha) \leftarrow \mathrm{update}(\mathbf{V}_h, \beta)$
    end if
end for

Section 3

Nakao's method with nonconforming elements

Subsections

Idea'

Let $X_{h0} \not\subset H_0^1(\Omega)$ be a nonconforming FE space and $u_h \in X_{h0}$ be a numerical solution obtained by FEM.

- We expect there exists an exact solution $u \in H_0^1$ nearby $u_h \in X_{h0}$.
- We want to verify existence of an exact solution in a "candidate" set U. We use Schauder's fixpoint theorem to verify the existence.
- For applying the fixpoint theorem, we now formulate sufficient conditions which we can check rigorously in computers.

$\Longrightarrow$ We now consider a fixpoint problem on $X_{h0} + H_0^1(\Omega)$.

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Definition. Candidate set' (type-I)

Let $U \subset X_{h0} + H_0^1(\Omega)$ be a closed convex set s.t.:

$$U = u_h + U_h + U_*,$$

$$U_h = \sum_{i=1}^{N_{h0}} U_i \varphi_i \subset X_{h0} \qquad (U_i \in \mathbf{IR},\ \mathbf{U}_h = {}^t(U_i)_i),$$

$$U_* = \left\{ u_* \in X_{h0}^{\perp} \;\middle|\; \|u\|_h \leq \alpha \right\} \subset X_{h0}^{\perp} \qquad (\alpha > 0).$$

We here call U a candidate set.

Definition. Candidate set' (type-II)

Let $U \subset X_{h0} + H_0^1(\Omega)$ be a closed convex set s.t.:

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$$U_h = \left\{ \sum_{i=1}^{N_{h0}} u_i \varphi_i \;\middle|\; {}^t(u_i)_i \in D^{-1} \mathbf{U}_h \right\} \qquad (\mathbf{U}_h \in \mathbf{IR}^{N_{h0}}),$$

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In a later slide, we will explain why we define the type-II.

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Let $P_h : X_{h0} + H_0^1(\Omega) \to X_{h0}$ be the orthogonal projection onto X_{h0}.

Infinite dimensional part + finite dimensional part

$-\triangle^{-1} f(U) \subset U$ holds if,

$$-(I - P_h)\triangle^{-1} f(U) \subset U_*, \quad \text{(A')}$$

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Let Π_h^{nc} be the nonconforming $\mathcal{P}_1$ interpolator from $H^2(\Omega) \cap H_0^1(\Omega)$ onto X_{h0}. Then, (A') holds if,

$$\|I - \Pi_h^{\mathrm{nc}}\| \sup_{u \in U} \|f(u)\|_{L^2} \leq \alpha \,. \tag{3}$$

Remark.

- If we know an explicit upper bound $C(h)$ of $\|I - \Pi_h^{\mathrm{nc}}\|$ and one of $\sup\|f(u)\|$, we can check (3) rigorously in computers.

An explicit upper bound of $\|I - \Pi_h^{\mathrm{nc}}\|$ is shown by [4, Liu '09].

Sufficient condition for finite dimensional part' (type-I)

Theorem. Sufficient condition for finite dimensional part' (type-I)

Let $D_{ij} := (\varphi_j, \varphi_i)_h$. Suppose interval vectors $\mathbf{d} = {}^t(d_i)_i \in \mathbf{IR}^{N_{h0}}$ and $\mathbf{V}_h \in \mathbf{IR}^{N_{h0}}$ satisfy $D^{-1}\mathbf{d} \subset \mathbf{V}_h$ and

$$\left\{ (f(u), \varphi_i)_{L^2} - (u_h, \varphi_i)_h + b(-\triangle^{-1} f(u) ; \varphi_i) \;\middle|\; u \in U \right\} \subset d_i \qquad (i = 1, \ldots, N_{h0}),$$

where $b(\,\cdot\,; \varphi_i)$ are boundary-integrations

$$b(v; \varphi_i) := \int_{\partial K_i} \frac{\partial v}{\partial \nu} \varphi_i \, \mathrm{d}s \qquad (v \in H^2(\Omega) \cap H_0^1(\Omega)).$$

Then, (B') holds if $\mathbf{V}_h \subset \mathbf{U}_h$.

Boundary-integration terms

Conforming case

- A FE function is an element of $H_0^1(\Omega)$. Therefore, the boundary-integration vanishes.
- Nakao's Theory implicitly assumes this property [6, 11, 12, 10].

Nonconforming case

- A FE function does not vanish on the boundary of its support. Therefore, the boundary-integration does not vanish.
- Using Bramble-Hilbert Lemma, we get a **constructive upper bound estimate** of $O(h)$ of $\left|b\left(-\triangle^{-1} f(u)\,;\varphi_i\right)\right|$.

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Lemma. Bramble-Hilbert

For every $v \in H^2(\Omega)$,

$$\inf_{p \in P_1(\Omega)} \|v + p\|_{H^2} \leq C_{\mathrm{BH}}(\Omega) \, |v|_{H^2} \ ,$$

where

$$C_{\mathrm{P}}(\Omega) := \frac{\operatorname{diam} \Omega}{\pi} ,$$

$$C_{\mathrm{BH}}(\Omega) := \sqrt{C_{\mathrm{P}}^2 \left(C_{\mathrm{P}}^2 + 1 \right) + 1} .$$

Theorem. Estimates of boundary-integrations

For every $v \in H^2(\Omega) \cap H_0^1(\Omega)$,

$$|b(v;\varphi_i)| \leq C_{\mathrm{BH}}\left(\hat{K}_{k(i)}\right) h \left\| \hat{\varphi}_{k(i)} \right\|_{H^1(\hat{K}_{k(i)})} \left\| \triangle v \right\|_{L^2(\Omega)} .$$

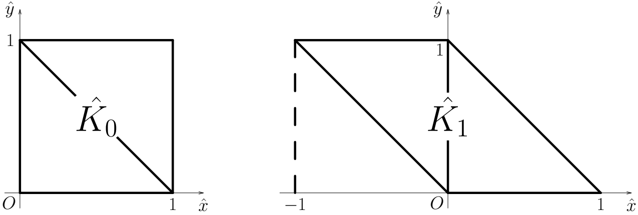

Figure: $\hat{K}_{k(i)} \sim K_i = \operatorname{supp} \varphi_i$

Numerical example with type-I

$$u := \pi^{-2}\sin(\pi x)\sin(\pi y)$$

$$\Longrightarrow \begin{cases} -\triangle u = 0.001\pi^2 u + 1.999\sin(\pi x)\sin(\pi y) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

# of iterations	candidate set U		next candidate set V		verification
	$\|\mathbf{U}_h\|_\infty$	α	$\|\mathbf{V}_h\|_\infty$	β	
1	1.0002e-5	0.1309620	11.1799	0.0795412	
2	11.1811	0.0796412	11.2103	0.0797180	
3	11.2114	0.0798180	11.2104	0.0797184	OK

Table: Verification with mesh size $h = 1/8$ and type-I candidate sets

- The interval vector $\mathbf{U}_h$ becomes too large.
- It is difficult to use for numerical validated computations.

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$$b(v; \varphi_i) := \int_{\partial K_i} \frac{\partial v}{\partial \nu} \varphi_i \,\mathrm{d}s \qquad (v \in H^2(\Omega) \cap H_0^1(\Omega)).$$

Then, (B') holds if $\mathbf{V}_h \subset \mathbf{U}_h$.

Sufficient condition for finite dimensional part' (type-II)

Theorem. Sufficient condition for finite dimensional part' (type-II)

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$$\left\{ (f(u), \varphi_i)_{L^2} - (u_h, \varphi_i)_h + b(-\triangle^{-1} f(u)\,; \varphi_i) \;\middle|\; u \in U \right\} \subset d_i \qquad (i = 1, \ldots, N_{h0}),$$

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$$b(v; \varphi_i) := \int_{\partial K_i} \frac{\partial v}{\partial \nu} \varphi_i \, \mathrm{d}s \qquad (v \in H^2(\Omega) \cap H^1_0(\Omega)).$$

Then, (B') holds if $\mathbf{d} \subset \mathbf{U}_h$.

We need NOT solve a linear system here!

Numerical example with type-I

$$u := \pi^{-2} \sin(\pi x) \sin(\pi y)$$

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Table: Verification with mesh size $h = 1/8$ and type-I candidate sets

Numerical example with type-II

$$u := \pi^{-2} \sin(\pi x) \sin(\pi y)$$

$$\Longrightarrow \begin{cases} -\triangle u = 0.001\pi^2 u + 1.999 \sin(\pi x) \sin(\pi y) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

# of iterations	candidate set U		next candidate set V		verification
	$\|\mathbf{U}_h\|_\infty$	α	$\|\mathbf{d}\|_\infty$	β	
1	0.00101012	0.02	0.89045	0.0795599	
2	0.899354	0.0895599	0.91856	0.0820849	
3	0.927745	0.0920849	0.919445	0.0821645	OK

Table: Verification with mesh size $h = 1/8$ and type-II candidate sets

Asymptotic behavior of a-posteriori error estimates

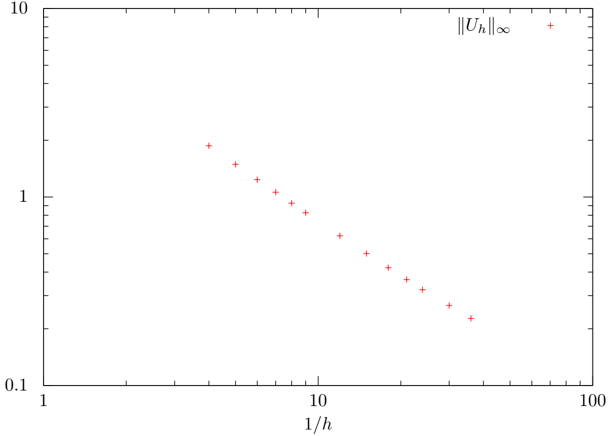

Figure: A-posteriori error esitimate of finite dimensional part

Asymptotic behavior of a-posteriori error estimates

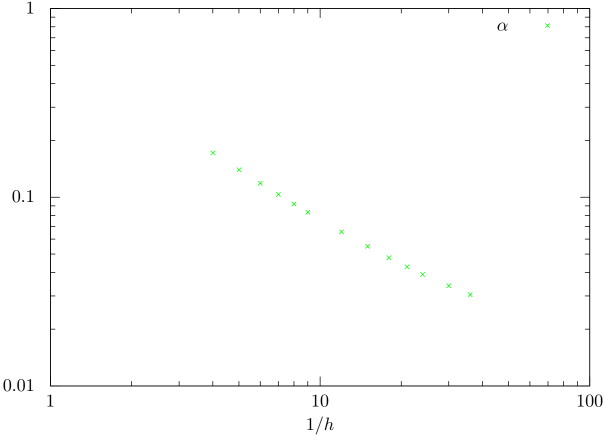

Figure: A-posteriori error esitimate of infinite dimensional part

Section 4

Summary

Summary

Our works

- generalized Nakao's method for nonconforming $\mathcal{P}_1$ FEM
- got constructive inequalities for boundary-integrations
- introduced type-II candidate sets
- showed numerically validated results for approximated solution to some elliptic BVP with nonconforming $\mathcal{P}_1$ FEM

Summary

Future works

- use other FEMs (e.g. $\mathcal{P}_2$ FEM)
- try on "non-classical" Nakao's methods
- apply our proposal method to other elliptic BVPs (e.g. 2D elasticity problem, free-boundary problem of vortex patch, etc.)

References I

Ivo Babuška and Manil Suri.
On locking and robustness in the finite element method.
SIAM J. Numer. Anal., 29(5):1261–1293, 1992.

Philippe G. Ciarlet.
The Finite Element Method for Elliptic Problems.
Society for Industrial and Applied Mathematics, 2002.

C.-O. Lee, J. Lee, and D. Sheen.
A locking-free nonconforming finite element method for planar linear elasticity.
Advances in Computational Mathematics, 19:277–291, July 2003.

References II

Xuefeng Liu.
Analysis of error constants for linear conforming and nonconforming finite elements.
PhD thesis, University of Tokyo, March 2009.

C. Lovadina.
A low-order nonconforming finite element for reissner-mindlin plates.
SIAM Journal on Numerical Analysis, 42(6):2688–2705, 2005.

References III

M. T. Nakao, N. Yamamoto, and K. Nagatou.
Numerical verifications for eigenvalues of second-order elliptic operators.
Japan Journal of Industrial and Applied Mathematics, 16(3):307–320, October 1999.

Mitsuhiro T. Nakao.
A numerical approach to the proof of existence of solutions for elliptic problems.
Japan Journal of Applied Mathematics, 5(2):313–332, 1988.

References IV

Siegfried M. Rump.
Validated solution of large linear systems.
Validation Numerics Computing Supplementum, 9:191–212, 1993.

Siegfried M. Rump.
A note on epsilon-inflation.
Reliable Computing, 4(4):371–375, November 1998.

Nobito Yamamoto and Mitsuhiro T. Nakao.
Numerical verifications for solutions to elliptic equations using residual iterations with a higher order finite element.
Journal of Computational and Applied Mathematics, 60(1–2):271–279, June 1995.

References V

中尾 充宏 and 渡部 善隆.
『実例で学ぶ精度保証付き数値計算 ～理論と実装 ～』.
臨時別冊・数理科学 2011 年 10 月. サイエンス社, 2011.

中尾 充宏 and 山本 野人.
『精度保証付き数値計算—コンピュータによる無限への挑戦』.
チュートリアル／応用数理の最前線. 日本評論社, 1998.

More precise fixpoint formulation

Let $f\colon L^2(\Omega) \to L^2(\Omega)$ be a map.

Assumption.

Assume, for any $h > 0$, $f|_{X_{h0}+H_0^1(\Omega)}$ is a continuous bounded map.

2D elliptic BVP

Find $u \in H^2(\Omega)$ s.t.

$$\begin{cases} -\triangle u = f(u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

More precise fixpoint formulation

Equivalent fixpoint problem

(*) Find $u \in H_0^1(\Omega)$ s.t.

$$u = -\triangle^{-1} f(u) \ .$$

Equivalent fixpoint problem on $X_{h0} + H_0^1$

(**) Find $u \in X_{h0} + H_0^1(\Omega)$ s.t.

$$u = -\triangle^{-1} f(u) \ .$$

More precise fixpoint formulation

Remark.

$$L^2 \xrightarrow{-\triangle^{-1}} H^2 \cap H_0^1 \hookrightarrow H_0^1 \hookrightarrow X_{h0} + H_0^1 \xrightarrow{f|_{X_{h0}+H_0^1}} L^2 .$$

- (Equiv.) If $u \in X_{h0} + H_0^1$ is a solution to (**), u is an element of H_0^1. Therefore, $u \in H_0^1$ is also a solution to (*).
- (Error) In general, a solution u is not an element of X_{h0}. However, we can measure an error between u and u_h in the $X_{h0} + H_0^1$ framework.

With type-II, do you really solve no linear systems?

In fact, one has to solve linear systems in other places.

Recall.

$$\left\{ (f(u), \varphi_i)_{L^2} - (u_h, \varphi_i)_h + b\big(-\triangle^{-1} f(u)\,; \varphi_i\big) \;\middle|\; u \in U \right\} \subset d_i\,,$$

$$\|I - \Pi_h^{\mathrm{nc}}\| \sup_{u \in U} \|f(u)\|_{L^2} \leq \alpha\,.$$

Sufficient condition for infinite dimensional part'

Theorem. Sufficient condition for infinite dimensional part'

Let $\mathit{\Pi}_h^{\mathrm{nc}}$ be the nonconforming $\mathcal{P}_1$ interpolator from $H^2(\Omega) \cap H_0^1(\Omega)$ onto X_{h0}. Then, (A') holds if,

$$\|I - \mathit{\Pi}_h^{\mathrm{nc}}\| \sup_{u \in U} \|f(u)\|_{L^2} \leq \alpha \,. \tag{3}$$

Remark.

- If we know an explicit upper bound $C(h)$ of $\|I - \mathit{\Pi}_h^{\mathrm{nc}}\|$ and one of $\sup\|f(u)\|$, we can check (3) rigorously in computers.

An explicit upper bound of $\|I - \mathit{\Pi}_h^{\mathrm{nc}}\|$ is shown by [4, Liu '09].

Proof. Sufficient condition for infinite dimensional part'

Assume $\|I - \Pi_h^{\mathrm{nc}}\| \sup_{u \in U} \|f(u)\|_{L^2} \le \alpha$. Let u be an element of the candidate set U. Then,

$$
\begin{aligned}
&\left\| -(I - P_h) \triangle^{-1} f(u) \right\|_h \\
&\le \|I - \Pi_h^{\mathrm{nc}}\| \left\| \triangle \triangle^{-1} f(u) \right\|_{L^2} \qquad \text{(by [4])} \\
&= \|I - \Pi_h^{\mathrm{nc}}\| \, \|f(u)\|_{L^2} \\
&\le \alpha \qquad \text{(by assumption)}.
\end{aligned}
$$

Consequently, $-(I - P_h) \triangle^{-1} f(u) \in U_*$ holds.

Sufficient condition for finite dimensional part' (type-I)

Theorem. Sufficient condition for finite dimensional part' (type-I)

Let $D_{ij} := (\varphi_j, \varphi_i)_h$. Suppose interval vectors $\mathbf{d} = {}^t(d_i)_i \in \mathbf{IR}^{N_{h0}}$ and $\mathbf{V}_h \in \mathbf{IR}^{N_{h0}}$ satisfy $D^{-1}\mathbf{d} \subset \mathbf{V}_h$ and

$$\left\{ (f(u), \varphi_i)_{L^2} - (u_h, \varphi_i)_h + b(-\triangle^{-1} f(u); \varphi_i) \;\middle|\; u \in U \right\} \subset d_i \qquad (i = 1, \ldots, N_{h0}),$$

where $b(\,\cdot\,; \varphi_i)$ are boundary-integrations

$$b(v; \varphi_i) := \int_{\partial K_i} \frac{\partial v}{\partial \nu} \varphi_i \, \mathrm{d}s \qquad (v \in H^2(\Omega) \cap H_0^1(\Omega)).$$

Then, (B') holds if $\mathbf{V}_h \subset \mathbf{U}_h$.

Proof. Sufficient condition for finite dimensional part'

Assume $\mathbf{V}_h \subset \mathbf{U}_h$. Let u be an arbitrary element of the candidate set U. Then, by definition of P_h, we get $-P_h \triangle^{-1} f(u) = u_h + v_h$ $(\exists v_h \in X_{h0})$. We denote $\mathbf{v}$ by the coefficient vector of v_h.

$$\begin{aligned}(v_h, \varphi_i)_h &= \left(-P_h \triangle^{-1} f(u) - u_h, \varphi_i\right)_h \\ &= \left(-\triangle^{-1} f(u), \varphi_i\right)_h - (u_h, \varphi_i)_h \\ &= (f(u), \varphi_i)_{L^2} + b\left(-\triangle^{-1} f(u)\,; \varphi_i\right) - (u_h, \varphi_i)_h \\ &\in d_i\,,\end{aligned}$$

$$\therefore D\mathbf{v} \in \mathbf{d}\,.$$

Therefore, by the assumption, it follows that $v_h \in U_h$.

Theorem. Estimates of boundary-integrations

For every $v \in H^2(\Omega) \cap H_0^1(\Omega)$,

$$|b(v;\varphi_i)| \le C_{\mathrm{BH}}\left(\hat{K}_{k(i)}\right) h \left\|\hat{\varphi}_{k(i)}\right\|_{H^1(\hat{K}_{k(i)})} \|\triangle v\|_{L^2(\Omega)} \, .$$

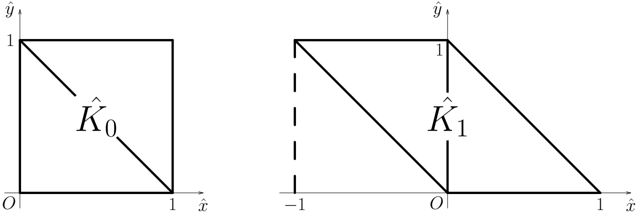

Figure: $\hat{K}_{k(i)} \sim K_i = \operatorname{supp} \varphi_i$

Proof. Estimates of boundary-integrations

Using change of variables by (the restriction of) the affine map $\Phi_i\colon K_i \xrightarrow{1:1} \hat{K}_k$, it follows from integration by parts:

$$|b(v;\varphi_i)| \leq \|\hat{v}\|_{H^2(\hat{K}_k)} \cdot \|\hat{\varphi}_k\|_{H^1(\hat{K}_k)} ,$$

$$\therefore \|b((\,\cdot\,)\circ\Phi_i;\varphi_i)\| \leq \|\hat{\varphi}_k\|_{H^1(\hat{K}_k)} .$$

Note that $b((\,\cdot\,)\circ\Phi_i;\varphi_i)$ vanishes on P_1. Then, applying Bramble-Hilbert lemma to $b((\,\cdot\,)\circ\Phi_i;\varphi_i)$, we get:

$$\begin{aligned} |b(v;\varphi_i)| &\leq C_{\mathrm{BH}}(\hat{K}_k)\,\|\hat{\varphi}_k\|_{H^1(\hat{K}_k)}\,|\hat{v}|_{H^2(\hat{K}_k)} \\ &= C_{\mathrm{BH}}(\hat{K}_k)\,\|\hat{\varphi}_k\|_{H^1(\hat{K}_k)}\,h|v|_{H^2(K_i)} . \end{aligned}$$