Some remarks on the rigorous estimation of inverse linear elliptic operators

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Aim

To verify the invertibility of a elliptic operator

$$\mathscr{L} := -\triangle + b\cdot\nabla + c : D(-\triangle) \to L^2(\Omega)$$

and compute $M > 0$ satisfying

$$\left\|\mathscr{L}^{-1}\right\|_{\mathscr{L}\left(L^2(\Omega),H_0^1(\Omega)\right)} \le M.$$

- $\Omega \subset \mathbb{R}^d$, $(d \in \{1,2,3\})$ bounded polygonal or polyhedral domain.
- $b \in L^\infty(\Omega)^d$, $c \in L^\infty(\Omega)$ are complex valued functions with norms:

$$\|b\|_{L^\infty(\Omega)^d} := \operatorname*{ess\,sup}_{x\in\Omega} \sqrt{|b_1(x)|^2+\cdots+|b_d(x)|^2}, \quad \|c\|_{L^\infty(\Omega)} := \operatorname*{ess\,sup}_{x\in\Omega} |c(x)|.$$

- $D(-\triangle) := \left\{u \in H_0^1(\Omega)\ ;\ -\triangle u \in L^2(\Omega)\right\}$ is a Banach space w.r.t. graph norm.
- The bilinear form related to $\mathscr{L}$ will not assume corecivity.

In this talk, we propose a numerical method to verify the invertibility of $\mathscr{L}$ to be based on the Fredholm operator theory.

Contents

Previous works

- The Plum method
  - The invertibility of $\mathscr{L}$ is based on the Homotopy method.

[1] M. Plum; Eigenvalue inclusions for second-order ordinary differential operators by a numerical homotopy method, *J. Appl. Math. Phys.(ZAMP)*, **41** (1990), 205–226.

- The Oishi method
  - The invertibility of $\mathscr{L}$ is based on the Fredholm operator theory.

[2] S. Oishi; Numerical verification of existence and inclusion of solutions for nonlinear operator equations, *Journal of Computational and Applied Mathematics*, **60**, no. 1-2 (1995), 171–185.

- The Nakao method
  - M not converge to $\left\|\mathscr{L}^{-1}\right\|$.

[3] M. T. Nakao, K. Hashimoto, and Y. Watanabe; A numerical method to verify the invertibility of linear elliptic operators with applications to nonlinear problems, *Computing*, **75** (2005), 1–14.

  - This talk: we consider the method using the Fredholm operator theory to this.
- The Watanabe method
  - M is expected to converge to $\left\|\mathscr{L}^{-1}\right\|$.

[4] Y. Watanabe, T. Kinoshita, and M. T. Nakao; A posteriori estimates of inverse operators for boundary value problems in linear elliptic partial differential equations, *Mathematics of Computation*, **82** (2013), 1543–1557.

Note: The Watanabe method is not necessarily superior to The Nakao method.

Projection and constants

- $S_h(\Omega)$: finite dimensional approximation subspace of $H_0^1(\Omega)$ dependent on the parameter $h > 0$.
- $n := \dim S_h(\Omega)$. $S_h(\Omega) = \text{span}\{\phi_1, \ldots, \phi_n\}$.
- $L_\phi \in \mathbb{C}^{n\times n}$, $L_{\phi,i,j} := (\phi_j, \phi_i)_{L^2(\Omega)}$, $\quad \forall i,j \in \{1,\ldots,n\}$.
- $D_\phi \in \mathbb{C}^{n\times n}$, $D_{\phi,i,j} := (\phi_j, \phi_i)_{H_0^1(\Omega)}$, $\quad \forall i,j \in \{1,\ldots,n\}$.
- $G_\phi \in \mathbb{C}^{n\times n}$, $G_{\phi,i,j} := (\nabla\phi_j, \nabla\phi_i)_{L^2} + ((b\cdot\nabla)\phi_j + c\phi_j, \phi_i)_{L^2}$, $\quad \forall i,j \in \{1,\ldots,n\}$.
- $\rho := \left\|D_\phi^{H/2} G_\phi^{-1} D_\phi^{1/2}\right\|_2$, $\quad \hat{\rho} := \left\|D_\phi^{H/2} G_\phi^{-1} L_\phi^{1/2}\right\|_2$.
- $P_h : H_0^1(\Omega) \to S_h(\Omega)$ denote the H_0^1-projection defined by

$$(u - P_h u, v_h)_{H_0^1(\Omega)} = 0, \qquad \forall v_h \in S_h.$$

- Suppose that P_h has the following approximation properties.

$$\|u - P_h u\|_{H_0^1(\Omega)} \le C(h)\, \|\triangle u\|_{L^2(\Omega)}, \qquad \forall u \in D(-\triangle)$$

$C(h) > 0$ is *numerically* determined with $C(h) \to 0$ as $h \to 0$.

Previous result [Computing, 2005]

Theorem 1 ([3, Theorem 2.1. & Theorem 2.2.])

If $\kappa := C(h)\big(\rho K(h) C_1 + C_2\big) < 1$ *then* $\mathscr{L}$ *is invertible and* M *is obtained by*

$$M = \frac{C_p}{1-\kappa}\left\|\begin{pmatrix}\rho\,(1 - C_2 C(h)) & \rho K(h)\\ \rho\, C_1 C(h) & 1\end{pmatrix}\right\|_2 \tag{1}$$

$$C_1 := \|b\|_{L^\infty} + C_p\,\|c\|_{L^\infty} \quad C_2 := \|b\|_{L^\infty} + C(h)\,\|c\|_{L^\infty} \quad \rho = \left\|D_\phi^{H/2} G_\phi^{-1} D_\phi^{1/2}\right\|_2$$

$$K(h) := \begin{cases} C(h)\left(C_p\|\nabla\cdot b\|_{L^\infty(\Omega)} + C_1\right), & \text{if} \quad b \in W^{1,\infty}(\Omega)^d \\ C_p C_2, & \text{if} \quad b \in L^\infty(\Omega)^d \end{cases}$$

It is expected $M \to \begin{cases} C_p\left\|\begin{pmatrix}\rho & 0\\ 0 & 1\end{pmatrix}\right\|_2 = C_p \max\{\rho, 1\}, & \text{if } b \in W^{1,\infty}(\Omega)^d \\ C_p\left\|\begin{pmatrix}\rho & \rho C_p\,\|b\|_{L^\infty}\\ 0 & 1\end{pmatrix}\right\|_2, & \text{if } b \in L^\infty(\Omega)^d \end{cases}$ as $h \to 0$.

Previous result [Math. Comp., 2013]

Theorem 2

If $\hat{\kappa} := C(h)C_2\left(\hat{\rho}C_1 + 1\right) < 1$ *then* $\mathscr{L}$ *is invertible and* M *is obtained by*

$$M = \frac{\sqrt{\hat{\rho}^2 + C(h)^2(1+\hat{\rho}C_1)^2}}{1-\hat{\kappa}} \tag{2}$$

It is expected $M \to \hat{\rho}$ as $h \to 0$.

$$C_1 := \|b\|_{L^\infty} + C_p\|c\|_{L^\infty}$$

$$C_2 := \|b\|_{L^\infty} + C(h)\|c\|_{L^\infty}$$

$$\hat{\rho} = \left\| D_\phi^{H/2} G_\phi^{-1} L_\phi^{1/2} \right\|_2$$

Remark: Let $\mathscr{L}_h^{-1} : L^2(\Omega) \to S_h(\Omega)$ be defined by $u_h := \mathscr{L}_h^{-1} f$ satisfying

$$(\nabla u_h, \nabla v_h)_{L^2} + ((b\cdot\nabla)u_h + cu_h, v_h)_{L^2} = (f, v_h)_{L^2}, \quad \forall v_h \in S_h(\Omega).$$

$$\Longrightarrow \left\| \mathscr{L}_h^{-1} \right\|_{\mathscr{L}\left(L^2(\Omega), H_0^1(\Omega)\right)} = \hat{\rho}.$$

Therefore, $\hat{\rho} = \left\|\mathscr{L}_h^{-1}\right\|_{\mathscr{L}\left(L^2(\Omega),H_0^1(\Omega)\right)} \to \left\|\mathscr{L}^{-1}\right\|_{\mathscr{L}\left(L^2(\Omega),H_0^1(\Omega)\right)}$ as $h \to 0$.

A consideration of these theorems

- It is expected that, as $h \to 0$,
  - Theorem 1: $M \to C_p \max\{\rho, 1\}$, ($b$:smooth)
  - Theorem 2: $M \to \hat{\rho}$

  which shows M by Theorem 2 could converge to the exact operator norm for $\mathscr{L}^{-1}$.
- In fact, $\hat{\rho} \le C_p \rho$ holds.
- Therefore, Theorem 2 looks more effective than Theorem 1.
- However, from the actual computation, since the criterion $\hat{\kappa} < 1$ is sometimes harder than $\kappa < 1$ for fixed h experimentally.

One-dimensional operator

$$\mathscr{L} := -\frac{d^2}{dx^2} + b\frac{d}{dx} + c : D(d^2/dx^2) \to L^2(\Omega)$$

- INTLAB Version 7 with MATLAB 8.0.0.783 (R2012b)
- $b(x) = r\sin(\pi x), \quad r, c \in \mathbb{R}, \quad \Omega := (0,1)$
- $S_h(\Omega)$ is hull of piecewise linear functions on equal partition $h > 0$
- $C(h) = h/\pi$, $C_p = 1/\pi$

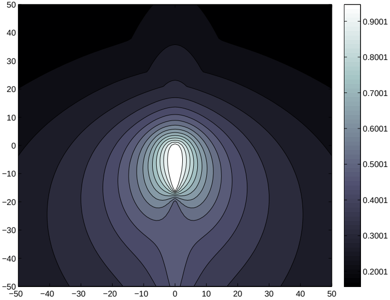

Figure : Distribution of $\hat{\rho}/(C_p\rho)$ for r (horizontal) and c (vertical)

One-dimensional operator

$$\mathscr{L} := -\frac{d^2}{dx^2} + b\frac{d}{dx} + c : D(d^2/dx^2) \to L^2(\Omega)$$

$b = 2.5\sin(\pi x), \quad c = -10$

$1/h$	ρ	$\hat{\rho}$	κ	$\hat{\kappa}$
10	12.6637	3.6970	0.6865	**1.9761**
30	12.9669	3.8003	0.0956	0.6249
50	12.9916	3.8084	0.0409	0.3696
100	13.0020	3.8119	0.0142	0.1827
200	13.0047	3.8128	0.0056	0.0908
500	13.0054	3.8130	0.0019	0.0362
1000	13.0055	3.8131	0.0009	0.0181

$b = -20\sin(\pi x), \quad c = -20$

$1/h$	ρ	$\hat{\rho}$	κ	$\hat{\kappa}$
10	2.6420	0.3552	**3.9293**	**6.8074**
30	2.5044	0.3542	0.5592	**2.2167**
50	2.4950	0.3542	0.2518	**1.3246**
100	2.4911	0.3542	0.0948	0.6603
200	2.4911	0.3542	0.0396	0.3296
500	2.4899	0.3542	0.0140	0.1318
1000	2.4899	0.3542	0.0067	0.0659

- The criterions of invertibility of $\mathscr{L}$, $\hat{\kappa} < 1$, is harder than $\kappa < 1$ in this example.
- The Nakao method [computing, 2005] has a meaning developed.

In this talk, we propose an alternative approach of Nakao's method.

Main result

Theorem 3

If $\kappa < 1$ then $\mathscr{L}$ is invertible and M is obtained by

$$M = \frac{\sqrt{\rho^2\left(C_p + C(h)(K(h) - C_pC_2)\right)^2 + C(h)^2\left(1 + \rho C_p C_1\right)^2}}{1 - \kappa} \tag{3}$$

It is expected $M \to C_p\rho$ as $h \to 0$.

$$C_1 = \|b\|_{L^\infty(\Omega)^d} + C_p\,\|c\|_{L^\infty(\Omega)}$$

$$C_2 = \|b\|_{L^\infty(\Omega)^d} + C(h)\,\|c\|_{L^\infty(\Omega)}$$

$$K(h) = \begin{cases} C(h)\left(C_p\|\nabla\cdot b\|_{L^\infty(\Omega)} + C_1\right), & \text{if} \quad b \in W^{1,\infty}(\Omega)^d \\ C_pC_2, & \text{if} \quad b \in L^\infty(\Omega)^d \end{cases}$$

$$\rho = \left\| D_\phi^{H/2} G_\phi^{-1} D_\phi^{1/2} \right\|_2$$

Note: The estimates of Theorem 1 is expected $M \to C_p\max\{\rho, 1\}$ as $h \to 0$.
Therefore, (3) succeeded removing $\max\{\cdot, 1\}$.

One-dimensional operator (1/5)

$$\mathscr{L} := -\frac{d^2}{dx^2} + b\frac{d}{dx} + c : D(d^2/dx^2) \to L^2(\Omega)$$

- INTLAB Version 7 with MATLAB 8.0.0.783 (R2012b)
- $b(x) = r\sin(\pi x), \quad r, c \in \mathbb{R}, \quad \Omega := (0,1)$
- $S_h(\Omega)$ is hull of piecewise linear functions on equal partition $h > 0$
- $C(h) = h/\pi$, $C_p = 1/\pi$

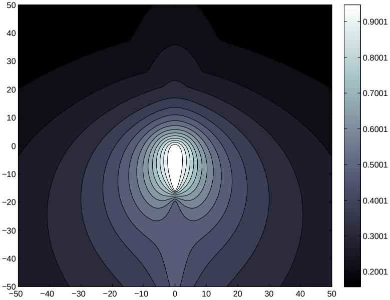

Figure : Distribution of $\hat{\rho}/(C_p\rho)$ for r (horizontal) and c (vertical)

One-dimensional operator (2/5)

$$\mathscr{L} := -\frac{d^2}{dx^2} + b\frac{d}{dx} + c : D(d^2/dx^2) \to L^2(\Omega)$$

$$b = 2.5\sin(\pi x), \quad c = -10$$

			Theorem 1		Theorem 3	Theorem 2	
$1/h$	ρ	$\hat{\rho}$	κ	M	M	$\hat{\kappa}$	M
10	12.6637	3.6970	0.6865	12.4285	**12.2786**	1.9761	——
30	12.9669	3.8003	0.0956	4.4655	**4.4598**	0.6249	10.1500
50	12.9916	3.8084	0.0409	4.2504	**4.2485**	0.3696	6.0452
100	13.0020	3.8119	0.0142	4.1667	**4.1663**	0.1827	4.6645
200	13.0047	3.8128	0.0056	4.1465	**4.1464**	0.0908	4.1936
500	13.0054	3.8130	0.0019	4.1409	4.1409	0.0362	**3.9561**
1000	13.0055	3.8131	0.0009	4.1401	4.1401	0.0181	**3.8832**

One-dimensional operator (3/5)

$$\mathscr{L} := -\frac{d^2}{dx^2} + b\frac{d}{dx} + c : D(d^2/dx^2) \to L^2(\Omega)$$

$$b = -20\sin(\pi x), \quad c = -20$$

			Theorem 1		Theorem 3	Theorem 2	
$1/h$	ρ	$\hat{\rho}$	κ	M	M	$\hat{\kappa}$	M
10	2.6420	0.3552	3.9293	——	——	6.8074	——
30	2.5044	0.3542	0.5592	1.8684	**1.5439**	2.2167	——
50	2.4950	0.3542	0.2518	1.0293	**0.9502**	1.3246	——
100	2.4911	0.3542	0.0948	0.8417	**0.8249**	0.6603	1.0469
200	2.4911	0.3542	0.0396	0.8040	0.8002	0.3296	**0.5289**
500	2.4899	0.3542	0.0140	0.7943	0.7938	0.1318	**0.4080**
1000	2.4899	0.3542	0.0067	0.7930	0.7929	0.0659	**0.3792**

One-dimensional operator (4/5)

$$\mathscr{L} := -\frac{d^2}{dx^2} + b\frac{d}{dx} + c : D(d^2/dx^2) \to L^2(\Omega)$$

$$b = \sin(\pi x), \quad c = 100$$

			Theorem 1		Theorem 3	Theorem 2	
$1/h$	ρ	$\hat{\rho}$	κ	M	M	$\hat{\kappa}$	M
10	0.9183	0.0500	1.1665	——	——	0.3516	**0.1508**
30	0.9911	0.0499	0.1458	0.4977	0.3920	0.0577	**0.0608**
50	0.9969	0.0499	0.0553	0.4060	0.3426	0.0275	**0.0542**
100	0.9992	0.0499	0.0155	0.3568	0.3242	0.0111	**0.0512**
200	0.9998	0.0499	0.0047	0.3365	0.3198	0.0049	**0.0504**
500	1.0000	0.0499	0.0012	0.3254	0.3186	0.0018	**0.0501**
1000	1.0000	0.0499	0.0005	0.3218	0.3184	0.0009	**0.0500**

One-dimensional operator (5/5)

$$\mathscr{L} := -\frac{d^2}{dx^2} + b\frac{d}{dx} + c : D(d^2/dx^2) \to L^2(\Omega)$$

$$b = \sin(\pi x), \quad c = -10$$

			Theorem 1		Theorem 3	Theorem 2	
$1/h$	ρ	$\hat{\rho}$	κ	M	M	$\hat{\kappa}$	M
10	94.9621	29.6261	2.1281	——	——	5.2424	——
30	231.4257	72.4346	0.5767	**172.3900**	172.4427	3.5678	——
50	261.5470	81.8835	0.2366	**108.4156**	108.4277	2.3262	——
100	276.7469	86.6517	0.0641	**93.8348**	93.8375	1.1938	——
200	280.8268	87.9316	0.0171	**90.7977**	90.7983	0.5964	217.8445
500	281.9909	88.2967	0.0032	**89.9844**	89.9846	0.2373	115.7653
1000	282.1580	88.3491	0.0010	**89.8696**	89.8697	0.1184	100.2071

Two-dimensional non-self-adjoint operator (1/3)

$$\mathscr{L} := -\triangle + b\cdot\nabla + c : D(-\triangle) \to L^2(\Omega)$$

$$b = R\begin{pmatrix} -y+1/2 \\ x-1/2 \end{pmatrix}, \quad R\in\mathbb{R}, \quad c\in\mathbb{C}, \qquad \Omega := (0,1)\times(0,1)$$

- linear and uniform triangular meshes on Ω with the element side length $h>0$ for a given finite element mesh
- $C(h) = 0.493h$ and $C_p = 1/(\pi\sqrt{2})$

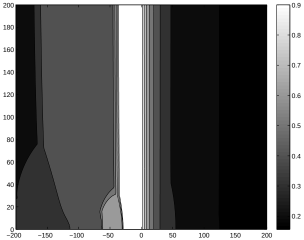

Figure : Distribution of $\hat{\rho}/(C_p\rho)$ for R (horizontal) and c (vertical)

Two-dimensional non-self-adjoint operator (2/3)

$$\mathscr{L} := -\triangle + b\cdot\nabla + c : D(-\triangle) \to L^2(\Omega)$$

$$b = R\begin{pmatrix} -y+1/2 \\ x-1/2 \end{pmatrix}, \quad R\in\mathbb{R}, \quad c\in\mathbb{C}, \qquad \Omega := (0,1)\times(0,1)$$

$$R = 10, \quad c = -10-5i$$

			Theorem 1		Theorem 3	Theorem 2	
$1/h$	ρ	$\hat{\rho}$	κ	M	M	$\hat{\kappa}$	M
5	1.7039	0.3656	2.3287	——	——	3.6305	——
10	1.7751	0.3946	0.7724	1.8734	**1.6510**	1.7974	——
20	1.7941	0.4025	0.2814	0.5384	**0.5033**	0.8798	3.4926
50	1.7995	0.4047	0.0869	0.4222	**0.4174**	0.3456	0.6227
100	1.8001	0.4050	0.0392	0.4092	**0.4082**	0.1716	0.4897
130	1.8004	0.4051	0.0294	0.4076	**0.4070**	0.1318	0.4670

Two-dimensional non-self-adjoint operator (3/3)

$$\mathscr{L} := -\triangle + b\cdot\nabla + c : D(-\triangle) \to L^2(\Omega)$$

$$b = R\begin{pmatrix} -y+1/2 \\ x-1/2 \end{pmatrix}, \quad R\in\mathbb{R}, \quad c\in\mathbb{C}, \qquad \Omega := (0,1)\times(0,1)$$

$$R = 10, \quad c = 15$$

			Theorem 1		Theorem 3	Theorem 2	
$1/h$	ρ	$\hat{\rho}$	κ	M	M	$\hat{\kappa}$	M
5	0.9732	0.1270	1.8758	——	——	1.9610	——
8	0.9903	0.1276	0.9032	3.3368	**2.6387**	1.1493	——
10	0.9939	0.1277	0.6488	0.8671	**0.6589**	0.8987	1.6951
20	0.9986	0.1279	0.2497	0.3543	0.2760	0.4284	**0.2453**
50	0.9999	0.1279	0.0818	0.2632	0.2316	0.1663	**0.1559**
100	1.0001	0.1279	0.0379	0.2426	0.2267	0.0823	**0.1400**

Linearized operator (1/2)

$$\mathscr{L} := -\triangle - 1 - 2u_h + 3au_h^2 : D(-\triangle) \to L^2(\Omega)$$

$\mathscr{L}$ is a linearized operator:

$$\begin{cases} -\triangle u = 1 + u + u^2 - au^3 & \text{in } \Omega := (0,1)\times(0,1), \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

at finite element solutions $u_h \in S_h(\Omega)$.

Lower approximate solution u_h for $a = 0.001$

			Theorem 1		Theorem 3	Theorem 2	
$1/h$	ρ	$\hat{\rho}$	κ	M	M	$\hat{\kappa}$	M
10	1.0586	0.2356	0.0030	0.2391	0.2447	0.0030	0.2421
20	1.0599	0.2379	0.0008	0.2388	0.2402	0.0008	0.2395
30	1.0601	0.2383	0.0004	0.2387	0.2394	0.0004	0.2391
40	1.0602	0.2385	0.0002	0.2387	0.2391	0.0002	0.2389
50	1.0603	0.2386	0.0002	0.2387	0.2389	0.0002	0.2388

$$\hat{\rho}/(C_p\rho) \sim 0.9995 \ (1/h = 50)$$

Linearized operator (2/2)

$$\mathscr{L} := -\triangle - 1 - 2u_h + 3au_h^2 : D(-\triangle) \to L^2(\Omega)$$

Upper approximate solution u_h for $a = 0.001$

			Theorem 1		Theorem 3	Theorem 2	
$1/h$	ρ	$\hat{\rho}$	κ	M	M	$\hat{\kappa}$	M
10	2.5948	0.3545	1.1823	——	——	0.7722	**1.9668**
20	2.6622	0.3624	0.2856	0.9204	0.8883	0.1861	**0.4756**
30	2.6758	0.3640	0.1262	0.7216	0.7074	0.0822	**0.4087**
40	2.6807	0.3645	0.0709	0.6671	0.6590	0.0461	**0.3887**
50	2.6830	0.3648	0.0453	0.6438	0.6386	0.0295	**0.3800**

$$\hat{\rho}/(C_p \rho) \sim 0.6040 \ (1/h = 50)$$

Conclusion

$$\mathscr{L} := -\triangle + b\cdot\nabla + c : D(-\triangle) \to L^2(\Omega)$$

- We propose a new computer-assisted procedure to compute a rigorous bound of the norm for $\mathscr{L}^{-1}$.
- The criterion for the invertibility of $\mathscr{L}$ is the same as Theorem 1, it has no limitation such that the lower bound of M is not less than 1.
- Although the proposed procedure would not converge to its exact operator norm, some verification examples show that it has a better bound than the approach in Theorem 2.
- We conclude that our proposed method should be a bridge the gap between the two previous approaches, and one may choice an appropriate procedure taking into consideration given problem or computational cost, and so on.