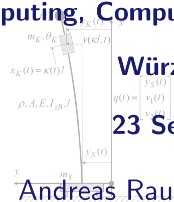
Universität Rostock Traditio et Innovatio

CoM Chair of Mechatronics

Exponential Enclosure Techniques for Initial Value Problems with Multiple Conjugate Complex Eigenvalues

SCAN 2014: 16th GAMM-IMACS International Symposium on Scientific Computing, Computer Arithmetic and Validated Numerics



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Contents

- Approaches for verified integration of ODEs in VALENCIA-IVP, alternatives to Taylor series or Taylor model-based techniques
- Fundamental iteration scheme
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- State-space transformation: Decoupling of state equations
- Use of complex-valued interval arithmetic
- Generalizations for systems with single and multiple eigenvalues and dominant (piecewise) linear dynamics
- Conclusions and outlook on future work

Initial Value Problem with Interval Uncertainty

Definition of the initial value problem (IVP)

- Given set of ordinary differential equations (ODEs)

$$\dot{\mathbf{x}}(t) = \mathbf{f}(\mathbf{x}(t))$$

  with smooth right-hand sides

- Uncertain initial conditions

$$\mathbf{x}(0) \in [\mathbf{x}_0] := [\mathbf{x}(0)] = [\underline{\mathbf{x}}(0) \; ; \; \overline{\mathbf{x}}(0)]$$

- Component-wise definition of interval vectors $[\mathbf{x}] = \begin{bmatrix}[x_1] & \dots & [x_n]\end{bmatrix}^T$ with the vector entries $[x_i] = [\underline{x}_i \; ; \; \overline{x}_i]$, $\underline{x}_i \leq x_i \leq \overline{x}_i$, $i = 1, \dots, n$

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$$\mathbf{x}(0) \in [\mathbf{x}_0] := [\mathbf{x}(0)] = [\underline{\mathbf{x}}(0)\,;\,\overline{\mathbf{x}}(0)]$$

- Uncertainty in parameters p_j, $j = 1, \ldots, n_p$, and control (input) signals $\mathbf{u}(t)$ are assumed to be included in the expression for $\mathbf{f}(\mathbf{x}(t))$, e.g. $\dot{p}_j = 0$ or $\mathbf{u}(t) = \mathbf{u}(\mathbf{x}(t), \mathbf{p})$

Fundamental Solution Approach

Definition of the state enclosure

$$\mathbf{x}^*(t) \in [\mathbf{x}](t) := \tilde{\mathbf{x}}(t) + [\mathbf{R}](t)$$

Constituents of the solution

- Approximate solution (non-verified) $\tilde{\mathbf{x}}(t)$
- Verified error bound $[\mathbf{R}](t)$
- Computation of the error bound by a suitable iteration scheme

Note

Without suitable counter-measures, solution enclosures may not converge, even for asymptotically stable systems

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Reference

Rauh, Andreas; Westphal, Ramona; Auer, Ekaterina; Aschemann, Harald: *Exponential Enclosure Techniques for the Computation of Guaranteed State Enclosures in* VALENCIA-IVP, Reliable Computing: Special volume devoted to material presented at SCAN 2012, Novosibirsk, Russia, Vol. 19, Issue 1, pp. 66-90, 2013.

Exponential Enclosure Technique

Definition of the state enclosure

- Representation of contracting state enclosures by using

$$\mathbf{x}^*(t) \in [\mathbf{x}_e](t) := \exp([\mathbf{\Lambda}] \cdot t) \cdot [\mathbf{x}_e](0)$$

with $0 \notin [x_{e,i}](0)$, $[\mathbf{x}_e](0) = [\mathbf{x}_0]$ for the diagonal matrix

$$[\mathbf{\Lambda}] := \operatorname{diag}\{[\lambda_i]\} \ , \quad i = 1, \ldots, n$$

with element-wise negative real entries λ_i

- Definition of the interval matrix exponential

$$\exp([\mathbf{\Lambda}] \cdot t) := \operatorname{diag}\{\exp([\lambda_1] \cdot t), \ldots, \exp([\lambda_n] \cdot t)\}$$

Exponential Enclosure Technique

Derivation of the iteration scheme

- Picard iteration

$$\mathbf{x}^*(t) \in [\mathbf{x}_e]^{(\kappa+1)}(t) := [\mathbf{x}_0] + \int_0^t \mathbf{f}\left([\mathbf{x}_e]^{(\kappa)}(s)\right) \mathrm{d}s$$

- Reformulation by the time-dependent expression

$$\mathbf{x}^*(t) \in \exp\left([\mathbf{\Lambda}]^{(\kappa+1)} \cdot t\right) \cdot [\mathbf{x}_e](0) = [\mathbf{x}_e]^{(\kappa+1)}(t)$$
$$=: [\mathbf{x}_0] + \int_0^t \mathbf{f}\left(\exp\left([\mathbf{\Lambda}]^{(\kappa)} \cdot s\right) \cdot [\mathbf{x}_e](0)\right) \mathrm{d}s$$

Exponential Enclosure Technique

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- Picard iteration

$$\mathbf{x}^*(t) \in [\mathbf{x}_e]^{(\kappa+1)}(t) := [\mathbf{x}_0] + \int_0^t \mathbf{f}\left([\mathbf{x}_e]^{(\kappa)}(s)\right) \mathrm{d}s$$

- Differentiation with respect to time and evaluation for $t \in [0\ ;\ T]$

$$\dot{\mathbf{x}}^*([0\ ;\ T]) \in \operatorname{diag}\left\{[\lambda_i]^{(\kappa+1)}\right\} \cdot \exp\left([\mathbf{\Lambda}]^{(\kappa+1)} \cdot [0\ ;\ T]\right) \cdot [\mathbf{x}_e](0)$$
$$\subseteq \mathbf{f}\left(\exp\left([\mathbf{\Lambda}]^{(\kappa)} \cdot [0\ ;\ T]\right) \cdot [\mathbf{x}_e](0)\right)$$

Exponential Enclosure Technique

Derivation of the iteration scheme

- Picard iteration

$$\mathbf{x}^*(t) \in [\mathbf{x}_e]^{(\kappa+1)}(t) := [\mathbf{x}_0] + \int_0^t \mathbf{f}\left([\mathbf{x}_e]^{(\kappa)}(s)\right) \mathrm{d}s$$

- Convergence of the iteration process for

$$\exp\left([\mathbf{\Lambda}]^{(\kappa+1)} \cdot t\right) \cdot [\mathbf{x}_e](0) \subseteq \exp\left([\mathbf{\Lambda}]^{(\kappa)} \cdot t\right) \cdot [\mathbf{x}_e](0)$$

equivalent to

$$[\lambda_i]^{(\kappa+1)} \subseteq [\lambda_i]^{(\kappa)} \qquad \text{and} \qquad [\mathbf{\Lambda}]^{(\kappa+1)} \subseteq [\mathbf{\Lambda}]^{(\kappa)}$$

Exponential Enclosure Technique

Derivation of the iteration scheme

- Picard iteration

$$\mathbf{x}^*(t) \in [\mathbf{x}_e]^{(\kappa+1)}(t) := [\mathbf{x}_0] + \int\limits_0^t \mathbf{f}\left([\mathbf{x}_e]^{(\kappa)}(s)\right) \mathrm{d}s$$

- Resulting iteration formula

$$[\lambda_i]^{(\kappa+1)} := \frac{f_i\left(\exp\left([\mathbf{\Lambda}]^{(\kappa)} \cdot [0\,;\,T]\right) \cdot [\mathbf{x}_e](0)\right)}{\exp\left([\lambda_i]^{(\kappa)} \cdot [0\,;\,T]\right) \cdot [x_{e,i}](0)}\,, \; i = 1, \ldots, n$$

with the guaranteed state enclosure at the point $t = T$

$$\mathbf{x}^*(T) \in [\mathbf{x}_e](T) := \exp\left([\mathbf{\Lambda}] \cdot T\right) \cdot [\mathbf{x}_e](0)$$

Exponential Enclosure Technique: Special Case

Application to linear system models

- Simplified state equations

$$f_i(\mathbf{x}) = \sum_{j=1}^{n} a_{ij} \cdot x_j$$

- Simplification of the iteration formula

$$[\lambda_i]^{(\kappa+1)} := \sum_{j=1, i \neq j}^{n} \left\{ a_{ij} \cdot \exp\left(\left([\lambda_j]^{(\kappa)} - [\lambda_i]^{(\kappa)} \right) \cdot [0\,;\ T] \right) \cdot \frac{[x_{e,j}](0)}{[x_{e,i}](0)} \right\}$$

$$+ a_{ii} \quad \text{with} \quad a_{ij} \in [a_{ij}]$$

Note

Free of overestimation if the equations are decoupled with $a_{ij} = 0$, $i \neq j$

Exponential Enclosure Technique: Special Case

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$$+ a_{ii} \quad \text{with} \quad a_{ij} \in [a_{ij}]$$

Solution

Transformation of the state equations (decoupling) into real-valued Jordan canonical form (assumption of pairwise different eigenvalues)

Exponential Enclosure Technique: Special Case

Application to linear system models

- Simplified state equations

$$f_i\left(\mathbf{x}\right) = \sum_{j=1}^{n} a_{ij} \cdot x_j$$

- Simplification of the iteration formula

$$[\lambda_i]^{(\kappa+1)} := \sum_{j=1, i\neq j}^{n} \left\{ a_{ij} \cdot \exp\left(\left([\lambda_j]^{(\kappa)} - [\lambda_i]^{(\kappa)} \right) \cdot [0\ ;\ T] \right) \cdot \frac{[x_{e,j}]\,(0)}{[x_{e,i}]\,(0)} \right\} + a_{ii} \quad \text{with} \quad a_{ij} \in [a_{ij}]$$

Computation of state transformation matrices

Use of approximately computed (floating point) eigenvector matrix with a verified inverse and a verified transformation of the initial states

Exponential Enclosures: State-Space Transformation

Decoupling of the state equations

$$\dot{\mathbf{z}}(t) = \mathbf{\Sigma} \cdot \mathbf{z}(t) \quad \text{with} \quad \mathbf{\Sigma} = \begin{bmatrix} \lambda_1 & 0 & \dots & 0 \\ 0 & \lambda_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \dots & 0 & \lambda_n \end{bmatrix} \quad \text{and} \quad \mathbf{z}(0) \in [\mathbf{z}(0)]$$

Applicability of the iteration scheme

The value zero must not be included in the true solution set
$\Longrightarrow$ Oscillating systems with complex eigenvalues are problematic

$$\mathbf{\Sigma} = \text{blkdiag}\{\dots, \bar{\mathbf{\Sigma}}_i, \dots\}\ , \quad \bar{\mathbf{\Sigma}}_i = \begin{bmatrix} \sigma_i & \omega_i \\ -\omega_i & \sigma_i \end{bmatrix}$$

Exponential Enclosures: State-Space Transformation

Decoupling of the state equations

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Solution: Use of complex valued Jordan canonical form

Complex-valued IVP $\dot{\mathbf{z}}(t) = \mathbf{\Sigma} \cdot \mathbf{z}(t)$ with $\mathbf{z}(0) \in \mathbb{C}^n$, $\mathbf{z}(0) \in [\mathbf{z}(0)]$

$$\mathbf{\Sigma} = \mathrm{blkdiag}\{\dots, \mathbf{\Sigma}_i, \dots\} \ , \quad \mathbf{\Sigma}_i = \begin{bmatrix} \sigma_i + \jmath\omega_i & 0 \\ 0 & \sigma_i - \jmath\omega_i \end{bmatrix}$$

Exponential Enclosures: State-Space Transformation

Decoupling of the state equations

$$\dot{\mathbf{z}}(t) = \mathbf{\Sigma} \cdot \mathbf{z}(t) \quad \text{with} \quad \mathbf{\Sigma} = \begin{bmatrix} \lambda_1 & 0 & \dots & 0 \\ 0 & \lambda_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \dots & 0 & \lambda_n \end{bmatrix} \quad \text{and} \quad \mathbf{z}(0) \in [\mathbf{z}(0)]$$

Analysis of the applicability: Eigenvalues of the multiplicity $\delta_i = 1$

- Exact solution $z_i(t) = e^{(\sigma_i + \jmath\omega_i)\cdot t} \cdot z_i(0)$, $z_{i+1}(t) = e^{(\sigma_i - \jmath\omega_i)\cdot t} \cdot z_{i+1}(0)$
- Iteration procedure is always applicable for $z_i(0) \neq 0$ due to

$$|z_i(t)|^2 = \left(e^{(\sigma_i + \jmath\omega_i)\cdot t} \cdot e^{(\sigma_i - \jmath\omega_i)\cdot t}\right) \cdot |z_i(0)|^2 = e^{2\sigma_i t} \cdot |z_i(0)|^2 \neq 0$$

Exponential Enclosures: State-Space Transformation

Decoupling of the state equations

$$\dot{\mathbf{z}}(t) = \mathbf{\Sigma} \cdot \mathbf{z}(t) \quad \text{with} \quad \mathbf{\Sigma} = \begin{bmatrix} \lambda_1 & 0 & \dots & 0 \\ 0 & \lambda_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \dots & 0 & \lambda_n \end{bmatrix} \quad \text{and} \quad \mathbf{z}(0) \in [\mathbf{z}(0)]$$

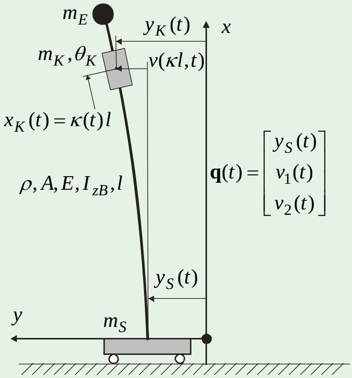
Analysis of the applicability: Eigenvalues of the multiplicity $\delta_i = 1$

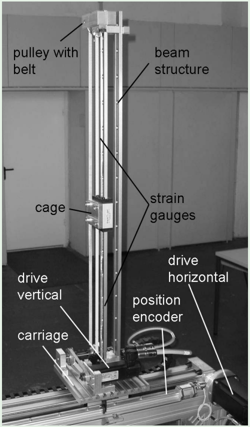
- Solution remains asymptotically stable for decoupled (oscillatory) linear systems
- (Limited) Overestimation in the initial conditions
- **Multiple eigenvalues lead to a non-negligible wrapping effect**

Simulation of the Dynamics of a Controlled High-Speed Rack Feeder System

Test rig at the Chair of Mechatronics, University of Rostock

Elastic multibody model





Simulation of the Dynamics of a Controlled High-Speed Rack Feeder System

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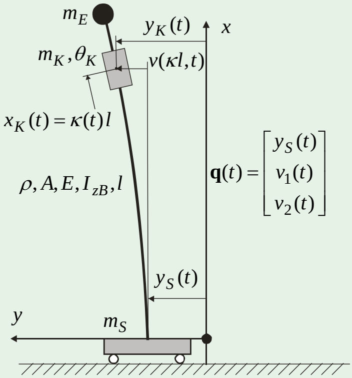
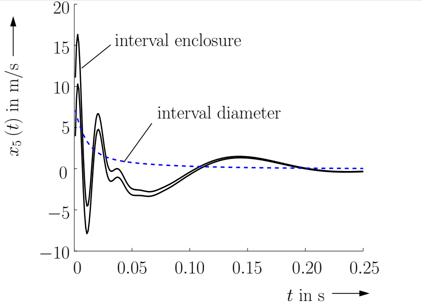


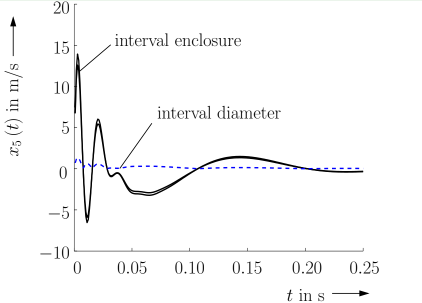
- Linear, time-invariant system model for $\kappa = \mathrm{const}$
- Nonlinear (resp. linear, time-varying) model for $\kappa \neq \mathrm{const}$
- System order 6 with real as well as complex eigenvalues of multiplicity 1
- Asymptotically stable after design of a suitable state feedback controller

Representative Simulation Results

Complex-valued exponential enclosure technique



VNODE-LP



- Increased diameters of the exponential state enclosures due to the wrapping effect in initial conditions: complex intervals are represented in midpoint/ radius form provided by INTLAB

Representative Simulation Results

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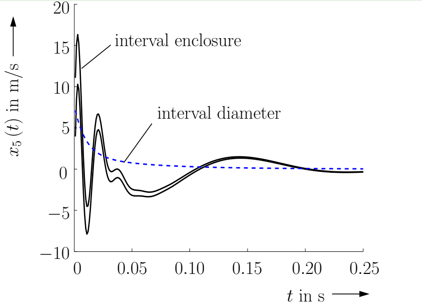

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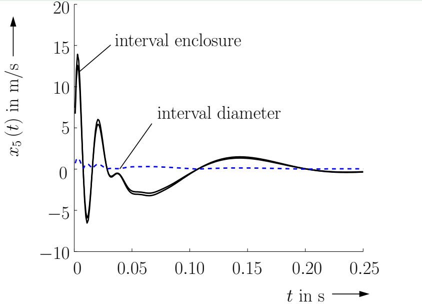
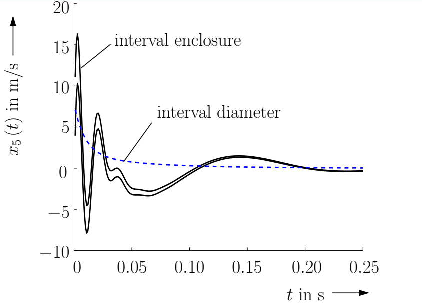


- Similar results can be obtained for the case of time-varying parameters κ with sufficiently small integration step sizes

Representative Simulation Results

Complex-valued exponential enclosure technique



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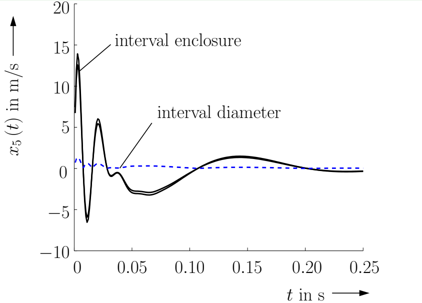

- A. Rauh, R. Westphal, H. Aschemann, Harald: *Verified Simulation of Control Systems with Interval Parameters Using an Exponential State Enclosure Technique*, Proc. of IEEE Intl. Conf. on Methods and Models in Automation and Robotics MMAR 2013, Miedzyzdroje, Poland, 2013.

Extension to Systems with Multiple (Complex) Eigenvalues

Canonical form with real eigenvalues

$$\dot{\mathbf{z}}(t) = \boldsymbol{\Sigma} \cdot \mathbf{z}(t) \quad \text{with} \quad \boldsymbol{\Sigma} = \mathrm{blkdiag}\{\lambda_1, \lambda_2 \ldots, \Sigma_i, \ldots \lambda_n\} \ ,$$

$$\boldsymbol{\Sigma}_i = \begin{bmatrix} \lambda_i & 1 & \ldots & 0 \\ 0 & \lambda_i & \ddots & \vdots \\ \vdots & \ddots & \ddots & 1 \\ 0 & \ldots & 0 & \lambda_i \end{bmatrix} \in \mathbb{R}^{\delta_i \times \delta_i} \quad \text{and} \quad \mathbf{z}(0) \in [\mathbf{z}(0)]$$

- Solve decoupled equations independently
- Jordan block with $\delta_i > 1 \Longrightarrow$ solve “from bottom to top”

Extension to Systems with Multiple (Complex) Eigenvalues

Canonical form with conjugate complex eigenvalues

$$\mathbf{\Sigma} = \mathrm{blkdiag}\{\dots, \mathbf{\Sigma}_i^+, \mathbf{\Sigma}_i^-, \dots\}$$

with $$\mathbf{\Sigma}_i^+ = \begin{bmatrix} \sigma_i + \jmath\omega_i & 1 & \dots & 0 \\ 0 & \sigma_i + \jmath\omega_i & \ddots & \vdots \\ \vdots & \ddots & \ddots & 1 \\ 0 & \dots & 0 & \sigma_i + \jmath\omega_i \end{bmatrix} \in \mathbb{C}^{\delta_i \times \delta_i}$$

and $$\mathbf{\Sigma}_i^- = \begin{bmatrix} \sigma_i - \jmath\omega_i & 1 & \dots & 0 \\ 0 & \sigma_i - \jmath\omega_i & \ddots & \vdots \\ \vdots & \ddots & \ddots & 1 \\ 0 & \dots & 0 & \sigma_i - \jmath\omega_i \end{bmatrix} \in \mathbb{C}^{\delta_i \times \delta_i}$$

for each eigenvalue pair $\sigma_i \pm \jmath\omega_i$ with $\delta_i > 1$

Extension to Systems with Multiple (Complex) Eigenvalues

Analytic representation of the solutions $z_{i+j}(t)$, $j = 0, \ldots, \delta_i - 1$

$$z^*_{i+j}(t) = e^{(\sigma_i + \jmath\omega_i)\cdot t} \cdot \left(\sum_{\zeta=j}^{\delta_i - 1} \frac{t^{\zeta - j}}{(\zeta - j)!} \cdot z_{i+\zeta}(0) \right)$$

Applicability of the standard exponential enclosure technique

- Computation of the square of its absolute value

$$|z^*_{i+j}(t)|^2 = e^{2\sigma_i \cdot t} \cdot \left| \left(\sum_{\zeta=j}^{\delta_i - 1} \frac{t^{\zeta - j}}{(\zeta - j)!} \cdot z_{i+\zeta}(0) \right) \right|^2 = e^{2\sigma_i \cdot t} \cdot |\chi_j|^2$$

- Exponential enclosure technique of VALENCIA-IVP is applicable if

$$0 \notin \left\{ \chi_j \middle| \chi_j = \Re\{\chi_j\} + \jmath\Im\{\chi_j\},\ z_{i+\zeta}(0) \in [z_{i+\zeta}(0)]\,,\ t \in [0\,;\,T] \right\}$$

holds for any $j = 0, \ldots, \delta_i - 1$

Extension to Systems with Multiple (Complex) Eigenvalues

Modification of the iteration scheme

- Definition of the enclosure and its time derivative

$$z_{i+j} = \left(\sum_{\zeta=j}^{\delta_i - 1} \frac{t^{\zeta-j}}{(\zeta-j)!} z_{i+\zeta}(0) \right) \cdot e^{\lambda_{i+j} t}$$

$$\dot{z}_{i+j} = \lambda_{i+j} \cdot \left(\sum_{\zeta=j}^{\delta_i - 1} \frac{t^{\zeta-j}}{(\zeta-j)!} z_{i+\zeta}(0) \right) \cdot e^{\lambda_{i+j} t} + \left(\sum_{\zeta=j+1}^{\delta_i - 1} \frac{t^{\zeta-(j+1)}}{(\zeta-(j+1))!} z_{i+\zeta}(0) \right) \cdot e^{\lambda_{i+j} t}$$

- Evaluation for $j = 0, \ldots, \delta_i - 1$, $z_\zeta(0) \in [z_\zeta(0)]$, $t \in [0\,;\,T]$
- Compute enclosures $[\lambda_i]$, ..., $[\lambda_{i+\delta_i-1}]$

Extension to Systems with Multiple (Complex) Eigenvalues

Modification of the iteration scheme

- Subsystem model (eigenvalue λ_i^* with multiplicity $\delta_i > 1$)

$$\begin{aligned}
\dot{z}_i &= \lambda_i^* \cdot z_i + z_{i+1} \\
\dot{z}_{i+1} &= \lambda_i^* \cdot z_{i+1} + z_{i+2} \\
&\vdots \\
\dot{z}_{i+\delta_i-1} &= \lambda_i^* \cdot z_{i+\delta_i-1}
\end{aligned}$$

- One-sided decoupling of equations
- Solutions can be computed in the order $z_{i+\delta_i-1}$, $z_{i+\delta_i-2}$, $\ldots$, z_{i+1}, z_i

Extension to Systems with Multiple (Complex) Eigenvalues

Iteration scheme

- Iteration procedure

$$[\lambda_{i+j}]^{(\kappa+1)} := \frac{\lambda_i^* \cdot \left(\sum_{\zeta=j}^{\delta_i - 1} \frac{t^{\zeta-j}}{(\zeta-j)!} z_{i+\zeta}(0) \right) \cdot e^{[\lambda_{i+j}]^{(\kappa)} t}}{\left(\sum_{\zeta=j}^{\delta_i - 1} \frac{t^{\zeta-j}}{(\zeta-j)!} z_{i+\zeta}(0) \right) \cdot e^{[\lambda_{i+j}]^{(\kappa)} t}} + \frac{\left(\sum_{\zeta=j+1}^{\delta_i - 1} \frac{t^{\zeta-(j+1)}}{(\zeta-(j+1))!} z_{i+\zeta}(0) \right) \cdot \left(e^{[\lambda_{i+j+1}]t} - e^{[\lambda_{i+j}]^{(\kappa)} t} \right)}{\left(\sum_{\zeta=j}^{\delta_i - 1} \frac{t^{\zeta-j}}{(\zeta-j)!} z_{i+\zeta}(0) \right) \cdot e^{[\lambda_{i+j}]^{(\kappa)} t}}$$

- One-sided decoupling: $[\lambda_{i+j}]$ depends on result for $[\lambda_{i+j+1}]$

Extension to Systems with Multiple (Complex) Eigenvalues

Iteration scheme

- Iteration procedure

$$[\lambda_{i+j}]^{(\kappa+1)} := \frac{\lambda_i^* \cdot \left(\sum_{\zeta=j}^{\delta_i - 1} \frac{t^{\zeta-j}}{(\zeta-j)!} z_{i+\zeta}(0) \right) \cdot e^{[\lambda_{i+j}]^{(\kappa)} t}}{\left(\sum_{\zeta=j}^{\delta_i - 1} \frac{t^{\zeta-j}}{(\zeta-j)!} z_{i+\zeta}(0) \right) \cdot e^{[\lambda_{i+j}]^{(\kappa)} t}} + \frac{\left(\sum_{\zeta=j+1}^{\delta_i - 1} \frac{t^{\zeta-(j+1)}}{(\zeta-(j+1))!} z_{i+\zeta}(0) \right) \cdot \left(e^{[\lambda_{i+j+1}] t} - e^{[\lambda_{i+j}]^{(\kappa)} t} \right)}{\left(\sum_{\zeta=j}^{\delta_i - 1} \frac{t^{\zeta-j}}{(\zeta-j)!} z_{i+\zeta}(0) \right) \cdot e^{[\lambda_{i+j}]^{(\kappa)} t}}$$

- Evaluation for $\lambda_i^* \in [\lambda_i^*]$, $z_\zeta(0) \in [z_\zeta(0)]$, $t \in [0\ ;\ T]$

Extension to Systems with Multiple (Complex) Eigenvalues

Iteration scheme

- Iteration procedure – simplified

$$[\lambda_{i+j}]^{(\kappa+1)} :=$$

$$\lambda_i^* + \frac{\left(\sum\limits_{\zeta=j+1}^{\delta_i-1} \frac{t^{\zeta-(j+1)}}{(\zeta-(j+1))!} z_{i+\zeta}(0)\right) \cdot \left(e^{\left([\lambda_{i+j+1}]-[\lambda_{i+j}]^{(\kappa)}\right)t} - 1\right)}{\left(\sum\limits_{\zeta=j}^{\delta_i-1} \frac{t^{\zeta-j}}{(\zeta-j)!} z_{i+\zeta}(0)\right)}$$

- Evaluation for $\lambda_i^* \in [\lambda_i^*]$, $z_\zeta(0) \in [z_\zeta(0)]$, $t \in [0\ ;\ T]$

Extension to Systems with Multiple (Complex) Eigenvalues

Practically important generalization

- If additional terms are included in the system model

$$\begin{aligned} \dot{z}_i &= \lambda_i^* \cdot z_i + z_{i+1} + f_i(\mathbf{z}) \\ \dot{z}_{i+1} &= \lambda_i^* \cdot z_{i+1} + z_{i+2} + f_{i+1}(\mathbf{z}) \\ &\vdots \\ \dot{z}_{i+\delta_i-1} &= \lambda_i^* \cdot z_{i+\delta_i-1} + f_{i+\delta_i-1}(\mathbf{z}) \ , \end{aligned}$$

a vector-valued iteration has to be performed

$$\begin{bmatrix} [\lambda_i]^{(\kappa+1)} \\ [\lambda_{i+1}]^{(\kappa+1)} \\ \vdots \\ [\lambda_{i+\delta_i-1}]^{(\kappa+1)} \end{bmatrix} \overset{!}{\subset} \begin{bmatrix} [\lambda_i]^{(\kappa)} \\ [\lambda_{i+1}]^{(\kappa)} \\ \vdots \\ [\lambda_{i+\delta_i-1}]^{(\kappa)} \end{bmatrix}$$

Extension to Systems with Multiple (Complex) Eigenvalues

Simplified enclosure

- Definition of the enclosure and its time derivative

$$z_{i+j} = (z_{i+j}(0) + t \cdot z_{i+j+1}(0)) \cdot e^{\lambda_{i+j} t}$$
$$\dot{z}_{i+j} = \lambda_{i+j} \left(z_{i+j}(0) + t \cdot z_{i+j+1}(0)\right) \cdot e^{\lambda_{i+j} t} + z_{i+j+1}(0) \cdot e^{\lambda_{i+j} t}$$

- Simplified iteration procedure

$$[\lambda_{i+j}]^{(\kappa+1)} := \frac{\lambda_i^* \cdot (z_{i+j}(0) + t \cdot z_{i+j+1}(0)) \cdot e^{[\lambda_{i+j}]^{(\kappa)} t}}{(z_{i+j}(0) + t \cdot z_{i+j+1}(0)) \cdot e^{[\lambda_{i+j}]^{(\kappa)} t}} + \frac{z_{i+j+1}(0) \cdot \left(e^{[\lambda_{i+j+1}] t} - e^{[\lambda_{i+j}]^{(\kappa)} t}\right)}{(z_{i+j}(0) + t \cdot z_{i+j+1}(0)) \cdot e^{[\lambda_{i+j}]^{(\kappa)} t}}$$

Extension to Systems with Multiple (Complex) Eigenvalues

Simplified enclosure

- Definition of the enclosure and its time derivative

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- Simplified iteration procedure

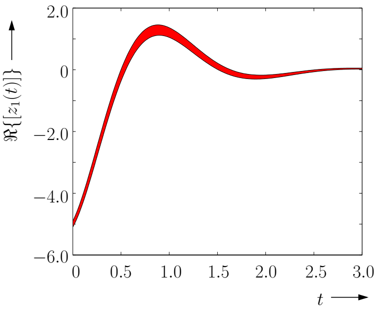
$$[\lambda_{i+j}]^{(\kappa+1)} := \lambda_i^* + \frac{e^{\left([\lambda_{i+j+1}]-[\lambda_{i+j}]^{(\kappa)}\right)t} - 1}{\frac{z_{i+j}(0)}{z_{i+j+1}(0)} + t}$$

- Evaluation for $j = 0, \ldots, \delta_i - 1$, $z_\zeta(0) \in [z_\zeta(0)]$, $t \in [0\,;\,T]$
- Decoupled/ coupled evaluation as before

Illustrative Example

Complex IVP with $\delta_i = 2$

$$\mathbf{z}(0) \in \begin{bmatrix} \langle -5, 0.1 \rangle \\ \langle -2, 0.1 \rangle \end{bmatrix} \ , \quad \lambda^* \in \langle -2 + 3\jmath, 0.1 \rangle$$



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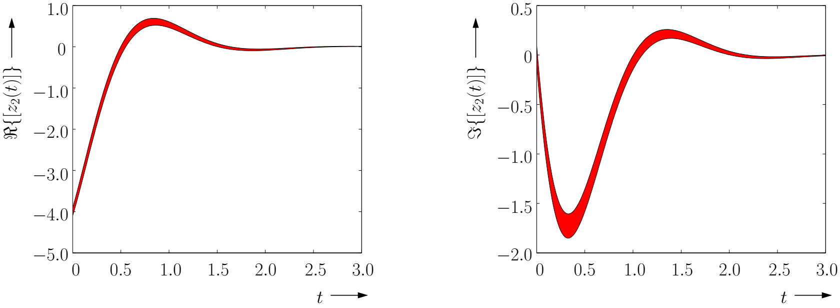

Conclusions and Outlook on Future Work

- Computationally efficient verified state enclosures for continuous-time dynamic systems with dominating linear dynamics
- Handling of uncertainty in initial conditions and parameters
- Iteration scheme based on complex-valued interval arithmetic
- Extensions to systems with multiple conjugate complex eigenvalues

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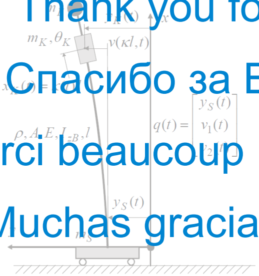
- Computationally efficient verified state enclosures for continuous-time dynamic systems with dominating linear dynamics
- Handling of uncertainty in initial conditions and parameters
- Iteration scheme based on complex-valued interval arithmetic
- Extensions to systems with multiple conjugate complex eigenvalues
- Verified, real-time capable safety analysis of control strategies: Verification of compatibility with state constraints
- Online sensitivity analysis in predictive control frameworks
- Online sensitivity analysis for state and parameter estimation
- Analysis of feedback linearizing control procedures
- Verification of interval-based sliding mode techniques



Dziękuję bardzo za uwagę!

Thank you for your attention!

Спасибо за Ваше внимание!

Merci beaucoup pour votre attention!

¡Muchas gracias por su atención!

Grazie mille per la vostra attenzione!

Vielen Dank für Ihre Aufmerksamkeit!