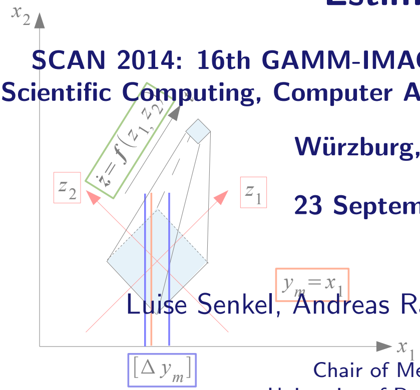
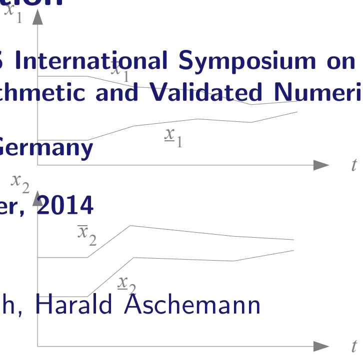
Universität Rostock Traditio et Innovatio

CoM Chair of Mechatronics

Computation of Confidence Regions in Reliable, Variable-Structure State and Parameter Estimation





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Chair of Mechatronics
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Contents

Motivation...

...for Calculating with Intervals in Control Tasks

- Guaranteed enclosures for parameters and states
- Dealing with uncertainty (caused by lack of knowledge about system parameters, inaccurate measurements, manufacturing tolerances, not exactly modeled effects, e.g. physical, mechanical)
- Quantification of worst-case influence of uncertainty on the system dynamics

Motivation...

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- Quantification of worst-case influence of uncertainty on the system dynamics

...for Using Sliding Mode Techniques

- Robustness despite uncertainty (unknown parameters, noise processes)
- Stabilization of error dynamics
- Guaranteed stability

This presentation...

...deals with

- Extension of an interval-based sliding mode observer for online state estimation and parameter identification
- Computation of guaranteed confidence regions for system states and parameters
- Matlab toolbox IntLab for interval computation

Properties of Interval Arithmetics

General

- Interval definition $[x] := [\underline{x}, \overline{x}] = [\inf([x]), \sup([x])]$
- Arithmetic operations $+$, $-$, $\cdot$, $/$ as for calculations with point-values
- Division by an interval containing zero is not allowed
- Inclusion monotonicity: split up interval boxes to reduce overestimation

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Problems

- Overestimation due to dependency problem
- Overestimation due to wrapping effect

Sliding Mode Techniques for State and Parameter Estimation

ODEs of a Dynamic System

$$\dot{\mathbf{x}}(t) = \mathbf{f}\left(\mathbf{x}(t), \mathbf{p}, \mathbf{u}(t)\right) = \mathbf{A} \cdot \mathbf{x}\left(t\right) + \mathbf{B} \cdot \mathbf{u}(t) + \mathbf{S} \cdot \xi\left(\mathbf{x}(t), \mathbf{u}(t)\right)$$
$$\mathbf{y}(t) = \mathbf{C} \cdot \mathbf{x}\left(t\right)$$

- Nominal expressions of system, input and output matrices $\mathbf{A} := \mathbf{A}(\mathbf{x}(t), \mathbf{p}) \in [\mathbb{A}]$, $\mathbf{B} := \mathbf{B}(\mathbf{x}(t), \mathbf{p}) \in [\mathbb{B}]$ and $\mathbf{C} := \mathbf{C}(\mathbf{x}(t), \mathbf{p}) \in [\mathbb{C}]$ (assumed to be included in interval expressions)
- State vector $\mathbf{x}(t)$, incl. uncertain/bounded parameters $\mathbf{p}(t) \in [\mathbf{p}]$
- Input vector $\mathbf{u}(t)$
- Representation of a-priori unknown and nonlinear terms $\mathbf{S} \cdot \xi\left(\mathbf{x}(t), \mathbf{u}(t)\right)$ with $\mathbf{S} \in \mathbb{R}^{n \times q}$ and $\|\xi\left(\mathbf{x}, \mathbf{u}\right)\| \leq \overline{\xi}$ (fixed upper bound of the vector norm $\overline{\xi}$)

Sliding Mode Techniques for State and Parameter Estimation

Sliding Mode Observer ODEs Considering Uncertainty $\hat{\tilde{\mathbf{f}}} := \hat{\tilde{\mathbf{f}}}(\hat{\mathbf{x}}(t), [\mathbf{p}], \mathbf{u}(t))$

$$\begin{aligned}\hat{\tilde{\mathbf{f}}} =& \hat{\mathbf{f}}(\hat{\mathbf{x}}(t), [\mathbf{p}], \mathbf{u}(t)) + \mathbf{P}^{+}[\hat{\mathbb{C}}]^{T} \cdot \mathbf{H}_s \cdot \mathsf{sign}(\mathbf{e}_m + [\Delta \mathbf{y}_m]) \\ :=& [\hat{\mathbb{A}}] \cdot \hat{\mathbf{x}}(t) + [\hat{\mathbb{B}}] \cdot \mathbf{u}(t) + \mathbf{H}_p \cdot [\mathbf{e}_m] + \mathbf{P}^{+}[\hat{\mathbb{C}}]^{T} \cdot \mathbf{H}_s \, \mathsf{sign}(\mathbf{e}_m + [\Delta \mathbf{y}_m])\end{aligned}$$

$$\hat{\mathbf{y}}_m := [\hat{\mathbb{C}}] \cdot \hat{\mathbf{x}}(t)$$

Sliding Mode Techniques for State and Parameter Estimation

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$$:= [\hat{\mathbb{A}}] \cdot \hat{\mathbf{x}}(t) + [\hat{\mathbb{B}}] \cdot \mathbf{u}(t) + \mathbf{H}_p \cdot [\mathbf{e}_m] + \mathbf{P}^{+}[\hat{\mathbb{C}}]^{T} \cdot \mathbf{H}_s \, \mathrm{sign}(\mathbf{e}_m + [\Delta \mathbf{y}_m])$$

$$\hat{\mathbf{y}}_m := [\hat{\mathbb{C}}] \cdot \hat{\mathbf{x}}(t)$$

- Combination of locally valid linear system model and variable structure part that handles uncertainty and nonlinearities to stabilize the error dynamics with certainty
- Uncertainty in parameters and measurements $\rightarrow$ interval arithmetic

Senkel, Luise; Rauh, Andreas; Aschemann, Harald: *Interval-Based Sliding Mode Observer Design for Nonlinear Systems with Bounded Measurement and Parameter Uncertainty*, IEEE Intl. Conference on Methods and Models in Automation and Robotics MMAR 2013, Miedzyzdroje, Poland, 2013.

Sliding Mode Techniques for State and Parameter Estimation

Sliding Mode Observer ODEs Considering Uncertainty

$$\hat{\tilde{\mathbf{f}}} := \hat{\tilde{\mathbf{f}}}(\hat{\mathbf{x}}(t), [\mathbf{p}], \mathbf{u}(t))$$

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$$\hat{\mathbf{y}}_m := [\hat{\mathbb{C}}] \cdot \hat{\mathbf{x}}(t)$$

- Instead of nominal system, input and output matrices $\rightarrow$ interval matrices $[\hat{\mathbb{A}}]$, $[\hat{\mathbb{B}}]$ and $[\hat{\mathbb{C}}]$ denoting the interval evaluations of $\hat{\mathbf{A}}(\hat{\mathbf{x}}(t), [\mathbf{p}]) \in [\hat{\mathbb{A}}]$, $\hat{\mathbf{B}}(\hat{\mathbf{x}}(t), [\mathbf{p}]) \in [\hat{\mathbb{B}}]$ and $\hat{\mathbf{C}}(\hat{\mathbf{x}}(t), [\mathbf{p}]) \in [\hat{\mathbb{C}}]$
- Measurement error vector $\mathbf{e}_m(t) \in [\mathbf{e}_m] = \mathbf{y}_m - \hat{\mathbf{y}}_m + [\Delta\mathbf{y}_m]$

Sliding Mode Techniques for State and Parameter Estimation

Sliding Mode Observer ODEs Considering Uncertainty

$$\hat{\tilde{\mathbf{f}}} := \hat{\tilde{\mathbf{f}}}(\hat{\mathbf{x}}(t), [\mathbf{p}], \mathbf{u}(t))$$

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- Underlying stabilization of the error dynamics by the observer gain matrix $\mathbf{H}_p$ (pole assignment, minimizing a quadratic cost function)
- Matrix $\mathbf{P}$ results from solving the Lyapunov equation $\tilde{\mathbf{A}} \cdot \mathbf{P} + \mathbf{P} \cdot \tilde{\mathbf{A}}^T + \mathbf{Q} = \mathbf{0}$ with $\tilde{\mathbf{A}} = \hat{\mathbf{A}} - \mathbf{H}_p \cdot \hat{\mathbf{C}}$ and $\mathbf{Q} > 0$
- Online evaluation of the switching amplitudes $\mathbf{H}_s$ in each time step to handle uncertainty

Sliding Mode Techniques for State and Parameter Estimation

Itô Differential Operator for Consideration of Stochastic Disturbances

$$L(V(t)) = \frac{\partial V}{\partial t} + \left(\frac{\partial V}{\partial \mathbf{e}}\right)^T \cdot \left(\mathbf{f}(\mathbf{x}, [\mathbf{p}], \mathbf{u}) - \hat{\tilde{\mathbf{f}}}(\hat{\mathbf{x}}, [\mathbf{p}], \mathbf{u})\right) + \frac{1}{2}\text{trace}\left\{\mathbf{G}^T \frac{\partial^2 V}{\partial \mathbf{e}^2} \mathbf{G}\right\}$$

- Suitable candidate of a Lyapunov function $V(t) = \frac{1}{2}(\mathbf{x} - \hat{\mathbf{x}})^T \mathbf{P}(\mathbf{x} - \hat{\mathbf{x}})$
- System $\mathbf{f}(\mathbf{x}, [\mathbf{p}], \mathbf{u})$ and observer parallel model $\hat{\tilde{\mathbf{f}}}(\hat{\mathbf{x}}, [\mathbf{p}], \mathbf{u})$
- Estimation error $\mathbf{e} = \mathbf{x} - \hat{\mathbf{x}}$
- Standard deviation of process and measurement noise $\mathbf{G} = [\mathbf{G}_p \quad -\mathbf{H}_p \mathbf{G}_m]$ to simulate neglected nonlinear phenomena or inaccurate sensor measurements

Sliding Mode Techniques for State and Parameter Estimation

Calculation of the Switching Amplitudes with $L(V(t)) \overset{!}{<} -\mathbf{q}^T \left|[\mathbf{e}_m]\right|$

$$\mathbf{h}_s = \begin{cases} \mathbf{0}, & \text{if } [\delta] \subseteq [\mathbf{e}_m]^T [\mathbf{e}_m] \\ \sup\left(\left|[\mathbf{e}_m]\right|^+ \cdot \left([\dot{V}_a] + \frac{1}{2} \cdot \text{trace}\left\{\mathbf{G}^T \frac{\partial^2 V}{\partial \mathbf{e}^2} \mathbf{G}\right\}\right) + \mathbf{q}^T\right), & \text{else} \end{cases}$$

- Element-wise non-negative defined stability margin $\mathbf{q} \geq \mathbf{0}$
- Small interval $[\delta]$ to prevent a division by zero
- $[\mathbf{e}_m] = \mathbf{y}_m(t) - \hat{\mathbf{y}}_m(t) + [\Delta \mathbf{y}_m]$
- $[\dot{V}_a] = [\mathbf{e}]^T \mathbf{P} \cdot ([\mathbf{f}] - [\hat{\mathbf{f}}] - \mathbf{H}_p \cdot \mathbf{e}_m)$ (time arguments are omitted)
- Matrix of switching amplitudes $\mathbf{H}_s = \text{diag}(\mathbf{h}_s) \in \mathbb{R}^{n_y \times n_y}$

Sliding Mode Techniques for State and Parameter Estimation

Calculation of the Switching Amplitudes with $L(V(t)) \overset{!}{<} -\mathbf{q}^T \left|[\mathbf{e}_m]\right|$

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- Absolute value of the difference between measured and estimated states $\left|[\mathbf{e}_m]\right|$ (component-wise)

$$\left|[e_{m,i}]\right| = \begin{cases} \left[-\overline{e}_{m,i}\ ;\ -\underline{e}_{m,i}\right] & \text{for } \overline{e}_{m,i} \leq 0 \\ \left[\underline{e}_{m,i}\ ;\ \overline{e}_{m,i}\right] & \text{for } \underline{e}_{m,i} \geq 0 \\ \left[0\ ;\ \max\{|\underline{e}_{m,i}|, |\overline{e}_{m,i}|\}\right] & \text{else} \ . \end{cases}$$

- Interval pseudo inverse $\left|[\mathbf{e}_m]\right|^{+} = \left(\left|[\mathbf{e}_m]\right|^T \left|[\mathbf{e}_m]\right| \right)^{-1} \cdot \left|[\mathbf{e}_m]\right|^T$

Sliding Mode Techniques for State and Parameter Estimation

Calculation of the Switching Amplitudes with $L(V(t)) \overset{!}{<} -\mathbf{q}^T \left|[\mathbf{e}_m]\right|$

$$\mathbf{h}_s = \begin{cases} \mathbf{0}, & \text{if } [\delta] \subseteq [\mathbf{e}_m]^T [\mathbf{e}_m] \\ \sup\left(\left|[\mathbf{e}_m]\right|^+ \cdot \left([\dot{V}_a] + \frac{1}{2} \cdot \text{trace}\left\{\mathbf{G}^T \frac{\partial^2 V}{\partial \mathbf{e}^2} \mathbf{G}\right\}\right) + \mathbf{q}^T\right), \text{else} \end{cases}$$

- Interval specifications for control, estimation and measurement errors acc. to $[\mathbf{e}] = [\mathbf{x}] - [\hat{\mathbf{x}}]$, $[\mathbf{x}] = \mathbf{x} + [\Delta\mathbf{x}_c]$, $[\hat{\mathbf{x}}] = \hat{\mathbf{x}} + [\Delta\mathbf{x}_e]$
- Stability proof (only in simulation) is successful, if $L(V(t)) < 0$, corresponding to $\dot{V}(t) < 0$

Simulation Example

Dynamics of a drive-train test-rig (system order $n = 2$)

$\dot{\mathbf{x}}(t) = \mathbf{f}(\mathbf{x}(t), [\mathbf{p}], \mathbf{u}(t)) = \begin{bmatrix} \dot{x}_1(t), & \dot{x}_2(t) \end{bmatrix}^T$ according to

$$\dot{\mathbf{x}}(t) = \mathbf{A} \cdot \mathbf{x}(t) + \mathbf{b} \cdot u(t) = \begin{bmatrix} 0 & 1 \\ 0 & \alpha \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} 0 \\ \beta \end{bmatrix} u(t)$$

$$y(t) = \mathbf{c}^T \cdot \mathbf{x} = [1 \quad 0] = x_1(t)$$

- Not a-priori known parameters $\alpha = -\frac{d}{J}$ and $\beta = \frac{1}{J}$, position x_1 measurable
- Mass moment of inertia J, velocity-proportional friction coefficient d

Simulation Example

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- Not a-priori known parameters $\alpha = -\frac{d}{J}$ and $\beta = \frac{1}{J}$, position x_1 measurable
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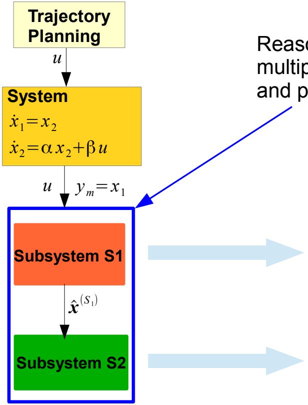
Aim

- Estimation of point-valued states x_1, x_2 (for state feedback control)
- Parameter identification of point-valued parameters $\hat{\alpha} \in [\alpha]$ and $\hat{\alpha} \in [\beta]$ ($[\alpha]$, $[\beta]$ — defined parameter intervals)
- Calculation of confidence regions of states $[\hat{x}_1]$ and $[\hat{x}_2]$ as well as parameters $[\hat{\alpha}]$ and $[\hat{\beta}]$

Simulation Example: Cascaded Structure

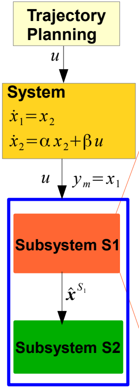


Reason for cascaded structure:
multiplicative coupling of states
and parameters in the system model

Estimation of system states
$$\hat{\boldsymbol{x}}^{(S_1)} = [\hat{x}_1 \quad \hat{x}_2 \quad \hat{\dot{x}}_2 \quad \hat{\ddot{x}}_2 \quad \hat{z}_1]^T$$

Identification of system parameters
$$\hat{\boldsymbol{x}}^{(S_2)} = [\hat{x}_2 \quad \hat{\dot{x}}_2 \quad \hat{\alpha} \quad \hat{\beta} \quad \hat{z}_2]^T$$

Simulation Example: Cascaded Structure



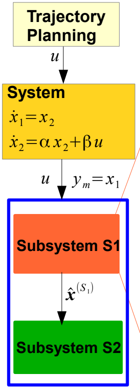
Model

$$\boldsymbol{f}^{(S1)} = \begin{bmatrix} x_2 + [\Delta x_{c,x_2,S1}] \\ \alpha(x_2 + [\Delta x_{c,x_2,S1}]) + \beta u \\ \alpha^2(x_2 + [\Delta x_{c,x_2,S1}]) + \alpha\beta u + \beta\dot{u} \\ \alpha^3(x_2 + [\Delta x_{c,x_2,S1}]) + \alpha^2\beta u + \alpha\beta\dot{u} + \beta\ddot{u} \\ \alpha^4(x_2 + [\Delta x_{c,x_2,S1}]) + \alpha^3\beta u + \alpha^2\beta\dot{u} + \alpha\beta\ddot{u} + \beta\dddot{u} \end{bmatrix}$$

$$y^{(S_1)} = x_1$$

Original model, 3 derivatives, model error

Simulation Example: Cascaded Structure



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$$y^{(S_1)} = x_1$$

Original model, 3 derivatives, model error

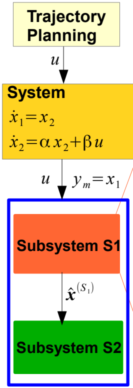
Observer parallel model

$$\hat{\boldsymbol{f}}^{(S_1)} = \begin{bmatrix} \hat{x}_2 + [\Delta x_{e,x_1,S1}] \\ \dot{\hat{x}}_2 + [\Delta x_{e,x_2,S1}] \\ \ddot{\hat{x}}_2 + [\Delta x_{e,x_3,S_1}] \\ \hat{z}_1 + [\Delta x_{e,x_4,S1}] \\ 0 + [\Delta x_{e,z,S1}] \end{bmatrix}$$

Integrator chain

Estimation of states: position, velocity, acceleration, jerk, model error of subsystem 1

Simulation Example: Cascaded Structure



Model

$$\boldsymbol{f}^{(S_1)} = \begin{bmatrix} x_2 + [\Delta x_{c,x_2,S1}] \\ \alpha(x_2 + [\Delta x_{c,x_2,S1}]) + \beta u \\ \alpha^2(x_2 + [\Delta x_{c,x_2,S1}]) + \alpha\beta u + \beta \dot u \\ \alpha^3(x_2 + [\Delta x_{c,x_2,S1}]) + \alpha^2\beta u + \alpha\beta \dot u + \beta \ddot u \\ \alpha^4(x_2 + [\Delta x_{c,x_2,S1}]) + \alpha^3\beta u + \alpha^2\beta \dot u + \alpha\beta \ddot u + \beta \ddot u \end{bmatrix}$$

$$y^{(S_1)} = x_1$$

Original model, 3 derivatives, model error

Observer parallel model

$$\hat{\boldsymbol{f}}^{(S_1)} = \begin{bmatrix} \hat x_2 + [\Delta x_{e,x_1,S1}] \\ \dot{\hat x}_2 + [\Delta x_{e,x_2,S1}] \\ \ddot{\hat x}_2 + [\Delta x_{e,x_3,S1}] \\ \hat z_1 + [\Delta x_{e,x_4,S1}] \\ 0 + [\Delta x_{e,z,S1}] \end{bmatrix}$$

Integrator chain

Interval-based
Sliding Mode Observer

$$\boldsymbol{A}^{(S_1)} = \begin{bmatrix} 0&1&0&0&0 \\ 0&0&1&0&0 \\ 0&0&0&1&0 \\ 0&0&0&0&1 \\ 0&0&0&0&0 \end{bmatrix} \qquad \boldsymbol{b}^{(S_1)} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\boldsymbol{e}_m^{(S_1)} = x_1 - \hat x_1$$

$$\dot{\hat{\boldsymbol{x}}}^{(S_1)} = \boldsymbol{A}^{(S_1)} \hat{\boldsymbol{x}}^{(S_1)} + \boldsymbol{b}^{(S_1)} u + \boldsymbol{h}_p^{(S_1)} \boldsymbol{e}_m^{(S_1)} + \mathrm{pinv}(\boldsymbol{P}^{(S_1)}) \; \boldsymbol{C}^{(S_1),T} \left(h_s^{(S_1)} \mathrm{sign}(e_m^{S_1}) \right)$$

Estimation of states: position, velocity, acceleration, jerk, model error of subsystem 1

Simulation Example: Cascaded Structure

- Similar procedure for subsystem S_2 with $\hat{\mathbf{x}}^{(S_2)} = \begin{bmatrix} \hat{x}_2 & \hat{x}_3 & \hat{\alpha}_e & \hat{\beta}_e & \hat{z}_2 \end{bmatrix}^T$
- Model $\mathbf{f}^{(S_2)}$, and observer parallel model $\hat{\mathbf{f}}^{(S_2)}$ with integrator disturbance models for the parameters according to

$$\dot{\alpha} = 0 + [\Delta x_{c,\alpha}], \quad \dot{\beta} = 0 + [\Delta x_{c,\beta}]$$
$$\dot{\hat{\alpha}} = 0 + [\Delta \alpha_\Delta], \quad \dot{\hat{\beta}} = 0 + [\Delta \beta_\Delta]$$

- Observer ODE:

$$\dot{\hat{\mathbf{x}}}^{(S_2)} = \mathbf{A}^{(S_2)}\hat{\mathbf{x}}^{(S_2)} + \mathbf{b}^{(S_2)}u$$
$$+\mathbf{H}_p^{(S_2)}\mathbf{e}_m^{(S_2)} + \mathrm{pinv}(\mathbf{P}^{(S_2)})\mathbf{C}^{(S_2),T} \cdot \mathbf{H}_s^{(S_2)}\mathrm{sign}([\mathbf{e}_m^{(S_2)}])$$

$\Rightarrow$ **significant influence of sign of $\dot{u}_S$ on observability in**

$$\mathbf{A}^{(S_2)} = \begin{bmatrix} 0 & 0 & x_{2,S} & 0 & 1 \\ \alpha_S^2 & 0 & 0 & \dot{u}_S & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \mathbf{b}^{(S_2)} = \begin{bmatrix} \beta_e \\ \alpha_e \cdot \beta_e \\ 0 \\ 0 \\ 0 \end{bmatrix}, \mathbf{C}^{(S_2)} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Calculation of Confidence Regions using Sliding Mode Techniques

Goal

Estimation of confidence regions for system states and parameters that represent all possible configurations in a guaranteed way

Possible Computational Strategies

- (1) Müller's Theorem (unstable solutions for S_2)
- (2) Calculation of enclosures for cooperative systems (unstable solutions for S_2)
- (3) Extension of (1) and (2): quasi-linear system representation by decoupling of system states $\Rightarrow$ generates a state-space transformation of the system into new coordinates $\Rightarrow$ recursive evaluation
- (4) Affine system representation

Preliminaries for Strategies (3) and (4)

Extension of Subsystem S_1

$$\dot{\hat{\mathbf{x}}}^{(S_1)} = \underbrace{\begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}}_{\mathbf{A}_O^{(S_1)}} \underbrace{\begin{bmatrix} \hat{x}_1 \\ \hat{x}_2 \\ \hat{x}_3 \\ \hat{x}_4 \\ \hat{z}_1 \end{bmatrix}}_{\hat{\mathbf{x}}^{(S_1)}} + \mathbf{H}_p^{(S_1)} \cdot (y_m - \hat{y}_m + [\Delta y_m])$$

$$+ \mathbf{P}_O^{(S_1)+} \mathbf{C}_O^{(S_1)} \mathbf{H}_s^{(S_1)} \cdot \mathsf{sign}(y_m - \hat{y}_m + [\Delta y_m])$$

Preliminaries for Strategies (3) and (4)

Extension of Subsystem S_1

Substitution of output equations $y_m = \mathbf{C}^{(S_1)}\mathbf{x}^{(S_1)}$, $\hat{y}_m = \mathbf{C}^{(S_1)}\hat{\mathbf{x}}^{(S_1)}$

$$\dot{\hat{\mathbf{x}}}^{(S_1)} = \underbrace{\begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}}_{\mathbf{A}_O^{(S_1)}} \underbrace{\begin{bmatrix} \hat{x}_1 \\ \hat{x}_2 \\ \hat{x}_3 \\ \hat{x}_4 \\ \hat{z}_1 \end{bmatrix}}_{\hat{\mathbf{x}}^{(S_1)}} + \mathbf{H}_p^{(S_1)}\mathbf{C}^{(S_1)} \cdot \left(\mathbf{x}^{(S_1)} - \hat{\mathbf{x}}^{(S_1)}\right) + \mathbf{H}_p^{(S_1)} \cdot [\Delta y_m]$$

$$+ \mathbf{P}_O^{(S_1)+}\mathbf{C}_O^{(S_1)}\mathbf{H}_s^{(S_1)} \cdot \mathrm{sign}(y_m - \hat{y}_m + [\Delta y_m])$$

Preliminaries for Strategies (3) and (4)

Extension of Subsystem S_1

Factor out the state vector

$$\dot{\hat{\mathbf{x}}}^{(S_1)} = \left(\mathbf{A}_O^{(S_1)} - \mathbf{H}_p^{(S_1)} \cdot \mathbf{C}^{(S_1)}\right) \underbrace{\begin{bmatrix} \hat{x}_1 \\ \hat{x}_2 \\ \hat{x}_3 \\ \hat{x}_4 \\ \hat{z}_1 \end{bmatrix}}_{\hat{\mathbf{x}}^{(S_1)}} + \mathbf{H}_p^{(S_1)} \mathbf{C}^{(S_1)} \cdot \left(\mathbf{x}^{(S_1)}\right) + \mathbf{H}_p^{(S_1)} \cdot [\Delta y_m]$$

$$+ \mathbf{P}_O^{(S_1)+} \mathbf{C}_O^{(S_1)} \mathbf{H}_s^{(S_1)} \cdot \mathsf{sign}\left(y_m - \hat{y}_m + [\Delta y_m]\right)$$

Preliminaries for Strategies (3) and (4)

Extension of Subsystem S_1

- Extension by integrator disturbance model for measurement interval
- Assumption: Measurement error interval is constant: $[\Delta \dot{y}_m] \equiv [0;0]$
- Reason: remove all additive interval offsets (6-D hyperbox of states)

$$\underbrace{\begin{bmatrix} \dot{\hat{x}}_1 \\ \dot{\hat{x}}_2 \\ \dot{\hat{x}}_3 \\ \dot{\hat{x}}_4 \\ \dot{\hat{z}}_1 \\ [\Delta \dot{y}_m] \end{bmatrix}}_{\dot{\hat{\mathbf{x}}}_{ext}^{(S_1)}} = \left(\underbrace{\begin{bmatrix} \mathbf{A}_O^{(S_1)} \\ \mathbf{0}^{1\times 6} \end{bmatrix} - \begin{bmatrix} \mathbf{H}_p^{(S_1)} \\ 0 \end{bmatrix} \cdot \begin{bmatrix} \mathbf{C}^{(S_1)} & 0 \end{bmatrix}}_{\mathbf{A}_{O,ext}^{(S_1)}} \right) \underbrace{\begin{bmatrix} \hat{x}_1 \\ \hat{x}_2 \\ \hat{x}_3 \\ \hat{x}_4 \\ \hat{z}_1 \\ [\Delta y_m] \end{bmatrix}}_{\hat{\mathbf{x}}_{ext}^{(S_1)}} + \mathbf{H}_p^{(S_1)} \underbrace{\mathbf{C}^{(S_1)} \cdot \left(\mathbf{x}^{(S_1)}\right)}_{y_m^{(S_1)}}$$

$$+ \underbrace{\mathbf{P}_O^{(S_1)+} \mathbf{C}_O^{(S_1)} \mathbf{H}_s^{(S_1)} \cdot \mathrm{sign}(y_m - \hat{y}_m + [\Delta y_m])}_{\text{point-value}}$$

Preliminaries for Strategies (3) and (4)

Extension of Subsystem S_2

Assumption: Measurement error interval is constant: $[\Delta \dot{y}_{m,i}] \equiv [0; 0]$

$$\underbrace{\begin{bmatrix} \dot{\hat{x}}_2 \\ \dot{\hat{x}}_3 \\ \dot{\hat{\alpha}} \\ \dot{\hat{\beta}} \\ \dot{\hat{z}}_2 \\ [\Delta \dot{y}_{m,1}] \\ [\Delta \dot{y}_{m,2}] \\ [\Delta \dot{y}_{m,1}] \end{bmatrix}}_{\dot{\hat{\mathbf{x}}}_{ext}^{(S_2)}} = \left(\underbrace{\begin{bmatrix} \mathbf{A}_O^{(S_2)} \\ \mathbf{0}^{3\times 8} \end{bmatrix} - \begin{bmatrix} \mathbf{H}_p^{(S_2)} \\ \mathbf{0}^{3\times 3} \end{bmatrix} \cdot \begin{bmatrix} \mathbf{C}^{(S_2)} & \mathbf{0}^{3\times 3} \end{bmatrix}}_{\mathbf{A}_{O,ext}^{(S_2)}} \right) \underbrace{\begin{bmatrix} \hat{x}_2 \\ \hat{x}_3 \\ \hat{\alpha} \\ \hat{\beta} \\ \hat{z}_2 \\ [\Delta y_{m,1}] \\ [\Delta y_{m,2}] \\ [\Delta y_{m,3}] \end{bmatrix}}_{\hat{\mathbf{x}}_{ext}^{(S_2)}}$$

$$+ \mathbf{H}_p^{(S_2)} \underbrace{\mathbf{C}^{(S_2)} \cdot (\mathbf{x}^{(S_2)})}_{\mathbf{y}_m^{(S_2)}} + \mathbf{P}_O^{(S_2)+} \mathbf{C}_O^{(S_2)} \mathbf{H}_s^{(S_2)} \cdot \mathrm{sign}(\mathbf{y}_m - \hat{\mathbf{y}}_m + [\Delta \mathbf{y}_m])$$

(3) State-Space Transformation and Recursive Evaluation

Aim

Reformulation of system ODEs such that the state enclosures are decoupled ⇒ Jordan matrices (real or complex eigenvalues at main diagonal), Metzler matrices

Problem of Interval Arithmetics

- Overestimation due to dependency problem: Occurrence of one interval several times in the same equation

$$[x] = [1;2]$$

$$f([x]) = [x] - [x] = [-1;1] \neq [0;0]$$

Figure: Dependency problem

(3) State-Space Transformation and Recursive Evaluation

Transformation $\mathbf{x}_{ext} = \mathbf{W}\mathbf{z}$ and $\dot{\mathbf{x}}_{ext} = \mathbf{W}\dot{\mathbf{z}}$

$$\dot{\mathbf{z}} = \mathrm{inv}(\mathbf{W})\dot{\mathbf{x}}_{ext}$$
$$\dot{\mathbf{z}} = \mathrm{inv}(\mathbf{W})(\mathbf{A}_{O,ext}\hat{\mathbf{x}}_{ext} + \mathbf{b}_{ext}u + \mathbf{H}_{p,ext}\mathbf{e}_m + \mathrm{pinv}(\mathbf{P}_{ext})\mathbf{C}_{ext}^T \cdot \mathbf{H}_s\mathrm{sign}([\mathbf{e}_m]))$$
$$\dot{\mathbf{z}} = \mathrm{inv}(\mathbf{W})(\mathbf{A}_{O,ext}\mathbf{W}\mathbf{z} + \mathbf{b}_{ext}u + \mathbf{H}_{p,ext}\mathbf{e}_m + \mathrm{pinv}(\mathbf{P}_{ext})\mathbf{C}_{ext}^T \cdot \mathbf{H}_s\mathrm{sign}([\mathbf{e}_m]))$$
$$\dot{\mathbf{z}} = \mathbf{J}\mathbf{z} + \mathrm{inv}(\mathbf{W})(\mathbf{b}_{ext}u + \mathbf{H}_{p,ext}\mathbf{e}_m + \mathrm{pinv}(\mathbf{P}_{ext})\mathbf{C}_{ext}^T \cdot \mathbf{H}_s\mathrm{sign}([\mathbf{e}_m]))$$

Definitions $(i = 1, ..., n)$

- Diagonal Matrix $\mathbf{J} = \mathrm{diag}(\lambda_i) = \mathrm{inv}(\mathbf{W})\mathbf{A}_{O,ext}\mathbf{W}$
- Matrix of eigenvectors $\mathbf{W}$
- (real or complex) eigenvalues $\lambda_i = \sigma_i \pm j \cdot \omega_i$
- Matlab command $[\mathbf{W}, \mathbf{J}] = \mathrm{eig}(\mathbf{A}_{ext} - \mathbf{H}_{p,ext} \cdot \mathbf{C}_{ext}^T)$
- $\mathbf{A}_{O,ext}$, $\mathbf{b}_{ext}$, $\mathbf{H}_{p,ext}$, $\mathbf{P}_{ext}$, $\mathbf{J}$, $\mathbf{W}$ separately for S_1 and S_2

(3) State-Space Transformation and Recursive Evaluation

Algorithm: Euler Method

- $[\mathbf{z}_k] = \mathrm{inv}(\mathbf{W}_k)\mathbf{W}_{k-1} \cdot [\mathbf{z}_{k-1}]$ (only for S_2)
- $\underline{\mathbf{z}}_k = \inf([\mathbf{z}_k])$ and $\underline{\mathbf{z}}_k = \sup([\mathbf{z}_k])$
- Lower bound: $\underline{\mathbf{z}}_{k+1} = \underline{\mathbf{z}}_k + T \cdot (\mathbf{J}_k \cdot \underline{\mathbf{z}}_k + \mathbf{b} \cdot u + \mathbf{H}_{p,k}^T \cdot \mathbf{y}_m$ $+\mathrm{pinv}(\mathbf{P}_k)\mathbf{C}^T \cdot \mathrm{sign}(\mathbf{y}_m - \hat{\mathbf{y}}_m + [\Delta\mathbf{y}_m]))$
- Upper bound: $\overline{\mathbf{z}}_{k+1} = \overline{\mathbf{z}}_k + T \cdot (\mathbf{J}_k \cdot \underline{\mathbf{z}}_k + \mathbf{b} \cdot u + \mathbf{H}_{p,k}^T \cdot \mathbf{y}_m$ $+\mathrm{pinv}(\mathbf{P}_k)\mathbf{C}^T \cdot \mathrm{sign}(\mathbf{y}_m - \hat{\mathbf{y}}_m + [\Delta\mathbf{y}_m]))$

$\rightarrow$ 4 cases because: $\mathbf{W}_k = \mathbf{W}_k(\mathrm{sign}(\dot{u}_k))$
$\rightarrow$ Method results for S_2 in unstable solutions due to wrapping effect caused by rotation of interval boxes (necessary because of switchings depending on $\dot{u} < 0$ or $\dot{u} > 0$)
$\rightarrow$ good results for S_1

(4) Affine System Representation

Aim

Recursion of state enclosures according to their constant initial values to reduce overestimation

Problems

- Overestimation due to wrapping effect: standard interval arithmetics only operate with axis-parallel boxes — wrapping of non-axis-parallel boxes by axis parallel ones

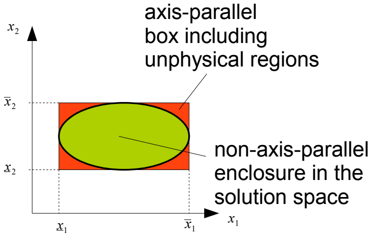


Figure: Wrapping effect

(4) Affine System Representation

Algorithm ($\dim(\mathbf{A}_{O,ext}^{(S_1)}) := n = 6$, $\dim(\mathbf{A}_{O,ext}^{(S_2)}) := n = 8$)

Initialization $\mathbf{M}_0 := \mathbf{I}^{n\times n}$, $\rho_0 := \mathbf{0}^{n\times 1}$
New system matrix of the discretized system $\mathbf{M}$
$[\mathbf{x}_{0,ext}]$ is the constant interval of initial values
For both subsystems separately:

$$\mathbf{M} = \mathbf{I}^{n\times n} + T \cdot \mathbf{A}_{O,ext}(t = t_k)$$

$$\mathbf{S}_k = \mathbf{P}_{ext} \cdot \mathbf{C}_{ext}^T \cdot \mathrm{diag}(\mathbf{h}_{s,k}) \cdot \mathrm{sign}(\mathbf{y}_{m,k} - \hat{\mathbf{y}}_{m,k} + [\Delta \mathbf{y}_m])$$

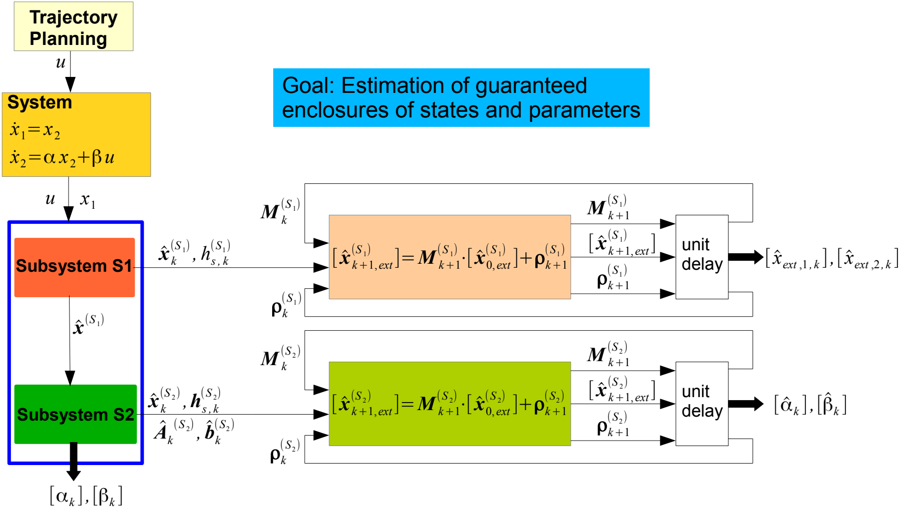
$$\mathbf{M}_{k+1} = \mathbf{M} \cdot \mathbf{M}_k$$

$$\rho_{k+1} = \mathbf{M}_k \cdot \rho_k + T \cdot (\mathbf{H}_{ext}^T \cdot \mathbf{y}_{m,k} + \mathbf{S}_k + \mathbf{b}_{k,ext} \cdot u_k)$$

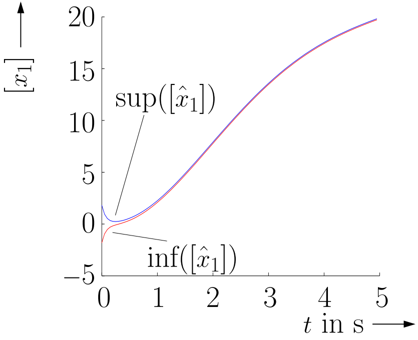
Update-Step:

$$[\mathbf{x}_{k+1,ext}] = \mathbf{M}_{k+1} \cdot [\mathbf{x}_{0,ext}] + \rho_{k+1}$$

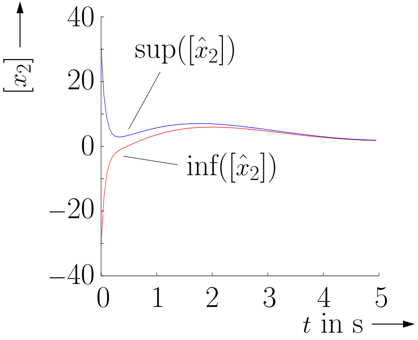
Summary


Trajectory Planning
System
$\dot{x}_1 = x_2$
$\dot{x}_2 = \alpha x_2 + \beta u$
Subsystem S1
Subsystem S2
$[\alpha_k], [\beta_k]$
Goal: Estimation of guaranteed enclosures of states and parameters
$[\hat{x}^{(S_1)}_{k+1,ext}] = M^{(S_1)}_{k+1} \cdot [\hat{x}^{(S_1)}_{0,ext}] + \rho^{(S_1)}_{k+1}$
$[\hat{x}^{(S_2)}_{k+1,ext}] = M^{(S_2)}_{k+1} \cdot [\hat{x}^{(S_2)}_{0,ext}] + \rho^{(S_2)}_{k+1}$
unit delay
$[\hat{x}_{ext,1,k}], [\hat{x}_{ext,2,k}]$
$[\hat{\alpha}_k], [\hat{\beta}_k]$

Results: State Estimation (Subsystem 1)



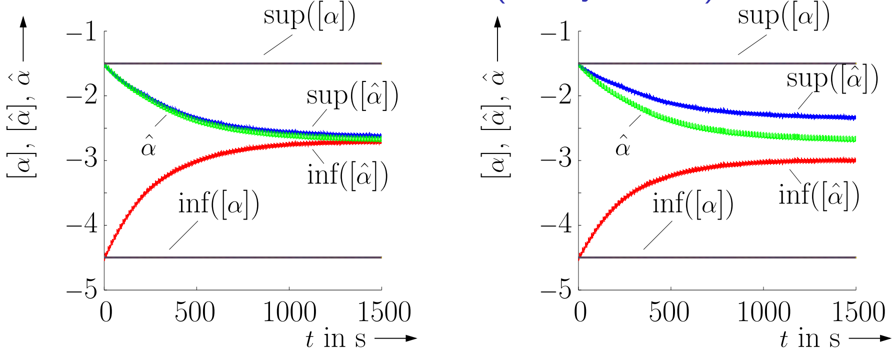
(a) Confidence region of $\hat{x}_1$.



(b) Confidence region of $\hat{x}_2$.

$\rightarrow$ Measurement error interval $[\Delta y_m] = [-0.05; 0.05]$

Results: Parameter Identification (Subsystem 2)



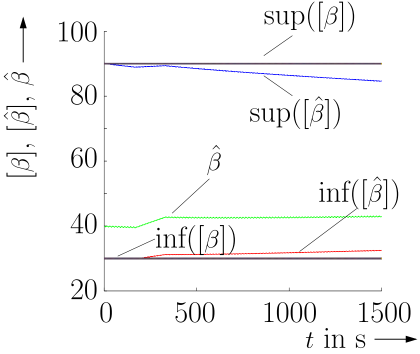
(c) Confidence region of $\hat{\alpha}$. (d) Confidence region of $\hat{\alpha}$.

left: $[\Delta y_{m,1}] = [-0.1; 0.1]$, $[\Delta y_{m,2}] = [-0.1; 0.1]$, $[\Delta y_{m,3}] = [-0.01; 0.01]$
$\mathbf{h}_{s,max} = 400$

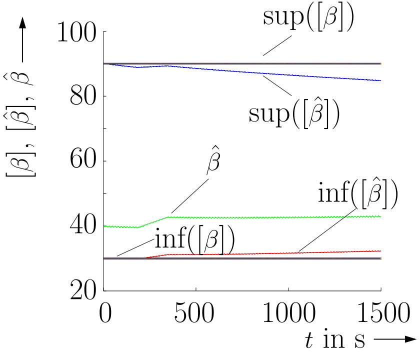
right: $[\Delta \mathbf{y}_{m,i}] = [-1; 1]$ with $i = 1, 2, 3$, $\mathbf{h}_{s,max} = 400$

$[\alpha]$, $[\beta]$ - defined intervals for stabilization of estimation error dynamics

Results: Parameter Identification (Subsystem 2)



(e) Confidence region of $\hat{\beta}$.



(f) Confidence region of $\hat{\beta}$.

left: $[\Delta y_{m,1}] = [-0.1; 0.1]$, $[\Delta y_{m,2}] = [-0.1; 0.1]$, $[\Delta y_{m,3}] = [-0.01; 0.01]$
$\mathbf{h}_{s,max} = 400$

right: $[\Delta \mathbf{y}_{m,i}] = [-1; 1]$ with $i = \{1, 2, 3\}$, $\mathbf{h}_{s,max} = 400$

$\rightarrow$ Smaller confidence region of β with multi-section strategy

Conclusions and Outlook

Conclusions

- Calculation of confidence regions by affine system representation
- Parallel estimation of point-values by interval-based Sliding Mode Observer

Conclusions and Outlook

Conclusions

- Calculation of confidence regions by affine system representation
- Parallel estimation of point-values by interval-based Sliding Mode Observer

Outlook

- Achievement of faster convergence for confidence regions by optimizing the system input (Pontrjagin's maximum principle, observability Gramian)
- Find automatized strategy to determine switching amplitude
- Application to other nonlinear systems

Thank you for your attention!

(1) Müller's Theorem

Aim

- Find lower and upper functions to bound the right-hand side of the system ODEs $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}(t), [\mathbf{p}], \mathbf{u}(t))$ that is affected by uncertain (interval) parameters $\mathbf{p} \in [\mathbf{p}]$

Lower function $\mathbf{v}(t)$, upper function $\mathbf{w}(t)$

- Worst-case enclosure by component-wise bounding system $v_i(t) \leq x_i(t) \leq w_i(t)$ for all $i = 1, ..., n$
- Determine differential inequalities $\dot{v}_i(t) \leq \dot{x}_i(t) \leq \dot{w}_i(t)$ for all possible states and parameters
- Solve a system of order $2n$

Solution unstable, too conservative enclosures (unphysical combinations), overestimation

(2) Calculation of Enclosures for Cooperative Systems

Definition

A given system $\mathbf{f}(\mathbf{x}(t), [\mathbf{p}], \mathbf{u}(t))$ is monotone concerning its initial values.

Extension of (1): Sufficient Conditions for a Cooperative System

- All states for all times are non-negative
  $\dot{x}_i = f_i(x_1, ..., x_{i-1}, 0, x_{i+1, ..., x_{n_x}}) \geq 0$
- All entries of the Jacobian fulfill $\frac{\partial f_i}{\partial x_j} \geq 0$ for all $i, j = \{1, ..., n_x\}$ $(i \neq j)$
- Calculation of guaranteed lower and upper bounds by evaluating the ODEs at all corner points instead of whole range of $x_i(t)$
  $\Rightarrow$ minimum and maximum value for all states for all time steps

Solution unstable, many configurations ($2^6 \cdot 6$ for TS1), also unphysical combinations possible, overestimation

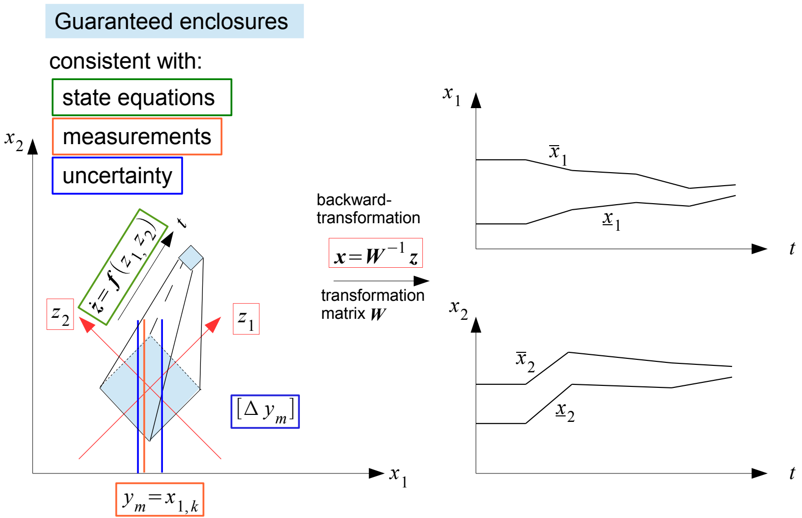


Figure: Principle of the State-Space-Transformation