Mathematically Rigorous Feasible Points

Some Observations on Exclusion Regions in Interval Branch and Bound Algorithms

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Outline

Overall Context

- In branch and bound (B&B) algorithms for constrained global optimization, an acceleration technique is to construct regions $\boldsymbol{x}^*$ around local optimizing points $\check{x}$, then delete these *exclusion regions* from further search.

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- $\boldsymbol{x}^*$ too large $\Longrightarrow$ solution bounds are not accurate.

History

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- Domes and Neumaier (March, 2014) propose a new technique for verification of feasibility, for construction of exclusion regions.

Scope of This Work

Contrast with Previous Work

- Previous work depended on uniqueness verification using Kuhn–Tucker conditions and interval Newton methods, or more sophisticated higher-order methods (in the case of pure nonlinear systems, especially.
- Motivation for this work is to imagine what conditions are necessary in a branch and bound algorithm for boxes adjacent to a small solution-containing box to be verified feasible, then construct the solution-containing box in such a way that adjacent boxes are sure to be verified infeasible. The exclusion region is then constructed to fit exactly this imagined set of adjacent boxes, eliminating the need for them to be created and processed by branch-and-bound.

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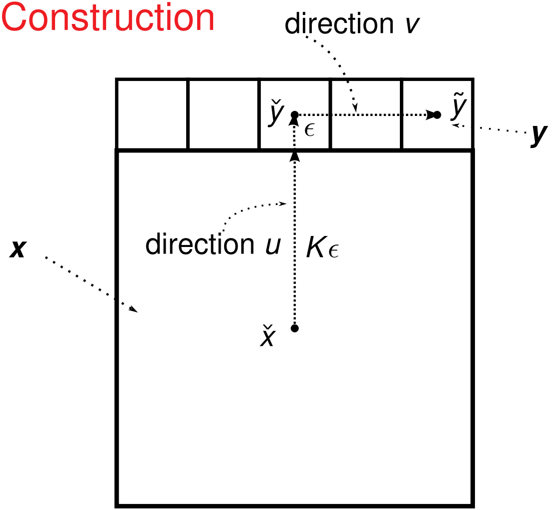
- We are also working on use of a third linear combination, for use in a rejection criterion.
- None of these is a catch-all to work on all problems.

Mathematically Rigorous Feasible Points

The Construction



For fixed ϵ, K is chosen so boxes $\boldsymbol{y}$ will certainly be rejected.

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- We subtract from that possible decrements in directions parallel to the box face.
- We bound the increments and decrements with interval evaluations, to prove the constraint is infeasible on a shell abutting the box.

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Problem Notation

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minimize $\varphi(x)$

subject to:

$$C(x) = (c_1(x), \ldots c_{m_1}(x)) = 0,$$

where $\varphi : \mathbb{R}^n \to \mathbb{R}$ and

$c_i : \mathbb{R}^n \to \mathbb{R}, i = 1, \ldots m_1$.

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The technique will work best if φ is simple, such as if φ consists of a single independent variable.

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- The idea is to maximize the rate of change in the direction u and minimize the rate of change in orthogonal directions.
- The heuristic we use here to achieve this is to choose that constraint combination

$$\tilde{c}_i = \sum_{j=1}^{m_1} \alpha_j c_j(x) \quad \text{which minimizes } \|(\nabla C(\check{x}))\alpha - e_i\|_2$$

over the parameters $\alpha = (\alpha_1, \ldots, \alpha_{m_1})$, where $\nabla C(\check{x})$ is the n by m_1 matrix whose i-th column is the gradient of c_i at $\check{x}$ and where $e_i \in \mathbb{R}^n$ is the i-th coordinate vector.

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Observations

- Minimizing $\|(\nabla C(\check{x}))\alpha - e_i\|_2$ is a floating point linear least squares problem, that can be done approximately.

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- Minimizing $\|(\nabla C(\check{x}))\alpha - e_i\|_2$ is a floating point linear least squares problem, that can be done approximately.
- High-quality library software for efficiently doing this is available.
- Once the optimal $\tilde{c}_i$ is obtain, we use it to compute an expansion factor K_i (depending on ϵ) for the i-th coordinate direction.

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- It is in general not possible to compute box width factors K_i for each coordinate direction, even with optimal $\tilde{c}_i$.

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Computing an Overall Expansion Factor $\overline{K}$

- It is in general not possible to compute box width factors K_i for each coordinate direction, even with optimal $\tilde{c}_i$.
- The boundary-box rejection criterion depends on each coordinate width, not just the i-th.
- We combine the expansion factors K_i of reasonable width to an overall expansion factor, used for every coordinate:

$$\overline{K} = \begin{cases} \displaystyle\max_{\substack{1\le i\le n, p\in\{-1,1\}\\ K_{i,p}<K_{\max}}} K_{i,p} & \text{if } \exists K_{i,p} < K_{\max}, \\ K_{\max} & \text{otherwise.} \end{cases}$$

$\overline{K}\epsilon$ will be the radius of the central box in each coordinate direction.

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- We tried the techniques on a subset of the COCONUT Lib-1 test set.
- We automatically converted the inequality constraints to equality constraints with bound constraints, for these problems.
- For each problem, we measured the number of faces of the central box upon which adjacent small boxes can be eliminated.

Experimental Results

- Of 26 problems tried, 15 had faces upon which adjacent small boxes could be eliminated.

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- Of the problems with box faces upon which adjacent small boxes could be eliminated, the number of faces ranged from $1/34$ to $32/44$.

Additional Observations

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- The linear combination of constraints is a kind of preconditioner for the system of constraints.
- We are presently investigating an alternate linear combination, motivated by the same ideas, for filtering boxes during the branch and bound algorithm.
- We are comparing the techniques, theoretically and in practice, to our optimally preconditioned interval Gauss–Seidel method.