A sharper error estimate of verified computations for nonlinear heat equations.

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Contents

1. Introduction and remark

Problem

Let Ω be a bounded and convex polygonal domain. We consider existence and uniqueness of the solution in the following Dirichlet type heat equations:

$$\begin{cases} \partial_t u - \Delta u = f(u) & \text{in } (0,\infty)\times\Omega, \\ u|_{\partial\Omega} = 0 & \text{on } (0,\infty)\times\partial\Omega, \\ u(0,x) = u_0(x) & \text{in } \Omega. \end{cases} \tag{1}$$

- Δ is represented Laplace operator.
- $u_0 \in H_0^1(\Omega)\cap H^2(\Omega)$.

Assumption

- $f : H_0^1(\Omega) \to L^2(\Omega)$ is a twice Fréchet differentiable nonlinear mapping
- There exists a non decreasing function $L_\rho > 0$ such that

$$\|f(u) - f(v)\|_{L^2(\Omega)} \leq L_\rho \rho \quad u, v \in U_\rho,$$

where for given $\rho > 0$, let

$$U_\rho := \{w \in H_0^1(\Omega) : \|w\|_{H_0^1} \leq \rho\}.$$

Remark

My abstruct is written in Scan2014, Book of Abstructs, p.119.

$$\Downarrow$$

We will talk shaper estimation of contents of the abstruct.

Essential estimate to obtain absolute error

According to the previous talk, to get rigorous estimate absolute error $\|\cdot\|_{L^\infty(T_1;H_0^1(\Omega))}$, we need to obtain sharp estimates:

$$\begin{cases} \delta := \left\| \int_{t_0}^t e^{-(t-s)\mathcal{A}}(\partial_s \omega(s) + \mathcal{A}\omega(s) - f(\omega))ds \right\|_{L^\infty\left(T_1;H_0^1(\Omega)\right)}, \\ \varepsilon := \|u(t_0) - \hat{u}_0\|_{H_0^1}. \end{cases}$$

We talk about a shaper estimations of δ by changing operator $\mathcal{A}$ and the semigroup $e^{-t\mathcal{A}}$ generated by $-\mathcal{A}$.

Difference form the previous talk

- Differential operator A (in this talk, A is represented $-\Delta$)

$$\mathcal{A} : H_0^1(\Omega) \to H^{-1}(\Omega) \quad \text{(Weak solution)}$$

$$\Downarrow$$

$$A : H_0^1(\Omega) \cap H^2(\Omega) \to L^2(\Omega) \quad \text{(Strong solution)}$$

- Semigroup

$$e^{-t\mathcal{A}} : H^{-1}(\Omega) \to H^{-1}(\Omega)$$

$$\Downarrow$$

$$e^{-tA} : L^2(\Omega) \to L^2(\Omega)$$

In the case of $\mathcal{A}$ (Weak solutions)

For fixed time t, from $\partial_t e^{-t\mathcal{A}} = \mathcal{A}e^{-t\mathcal{A}}$ holds, it sees that

$$\begin{aligned} & \left\| \int_0^t e^{-(t-s)\mathcal{A}} (\partial_s \omega(s) + \mathcal{A}\omega(s) - f(\omega(s))) ds \right\|_{H_0^1} \\ = & \left\| \int_0^t \mathcal{A} e^{-(t-s)\mathcal{A}} (\partial_s \omega(s) + \mathcal{A}\omega(s) - f(\omega(s))) ds \right\|_{H^{-1}} \\ = & \left\| \int_0^t \partial_s e^{-(t-s)\mathcal{A}} (\partial_s \omega(s) + \mathcal{A}\omega(s) - f(\omega(s))) ds \right\|_{H^{-1}} (2) \end{aligned}$$

(2) becomes a rough bound of δ.

In the case of A (Strong solutions)

For fixed time t, it sees that

$$
\begin{aligned}
&\int_0^t \|e^{-(t-s)A}(\partial_s\omega(s) + A\omega(s) - f(\omega(s)))\|_{H_0^1} ds \\
&\le \int_0^t \|A^{1/2}e^{-(t-s)A}(\partial_s\omega(s) + A\omega(s) - f(\omega(s)))\|_{L^2} ds \\
&\le \int_0^t e^{-1/2}(t-s)^{-1/2}\|(\partial_s\omega(s) + A\omega(s) - f(\omega(s)))\|_{L^2} ds \\
&\le 2e^{-1/2}t^{1/2}\|(\partial_s\omega + A\omega - f(\omega))\|_{L^\infty(T_1;L^2)} ds \\
&\approx O(\sqrt{\tau}). \qquad (3)
\end{aligned}
$$

2. Sharper estimation (Main theme)

Setting of approximate solutions

Let V_h be a finite dimensional subspace of $\mathcal{D}(A)$ and fix $\theta = 0, 1/2$ or 1[1] and $\hat{u}_{0,\theta} \in V_h$.
We employ the following full discretization schemes to obtain $u_{1,\theta} \in V_h$ such that for all $v_h \in V_h$,

$$\left(\frac{u^h_{1,\theta} - \hat{u}^h_{0,\theta}}{\tau}, v_h \right)_{L^2} + ((1-\theta)A\hat{u}^h_{0,\theta} + \theta A u^h_1, v_h)_{L^2} \\ = ((1-\theta)f(\hat{u}^h_0) + \theta f(u^h_1), v_h)_{L^2}. \quad (4)$$

We create $\omega_{0,\theta} \in L^\infty((t_0, t_1]; \mathcal{D}(A))$ by

$$\omega_{0,\theta}(t) := \hat{u}_{0,\theta}\phi_0(t) + \hat{u}_{1,\theta}\phi_1(t), \quad t \in (t_0, t_1],$$

where let $\hat{u}_{1,\theta} \in V_h$ be an approximation of u^h_1.

[1] Forward Euler ($\theta = 0$), Crank Nicolson ($\theta = 1/2$), Backward Euler ($\theta = 1$).

Main theme in this talk

For $\omega_{0,\theta}$ $(\theta = 0, 1$ or $1/2)$, we give the following estimation δ defined by

$$\left\| \int_{t_0}^{t} e^{-(t-s)A} (\partial_s \omega_{0,\theta}(s) + A\omega_{0,\theta}(s) - f(\omega_{0,\theta}(s))) ds \right\|_{L^\infty\left(T_1; H_0^1(\Omega)\right)} . \tag{5}$$

By estimating δ, we consider that which scheme gives shaper estimation of absolute error $\|u - \omega_{0,\theta}\|_{L^\infty(T_1; H_0^1)}$ in the case $\theta = 0, 1$ and $1/2$.

Estimation of δ

For given $0 \le \theta \le 1$, $\mathcal{C}_\theta \in L^2(\Omega)$ is defined by

$$\mathcal{C}_\theta := \frac{\hat{u}_{1,\theta} - \hat{u}_{0,\theta}}{\tau} + (1-\theta) A\hat{u}_{0,\theta} + \theta A\hat{u}_{1,\theta} - (1-\theta) f(\hat{u}_{0,\theta}) - \theta f(\hat{u}_{1,\theta}).$$

Let $g(t) := f(\hat{u}_{1,\theta})\phi_1(t) + f(\hat{u}_{0,\theta})\phi_0(t)$. We decompose

$$\left\| \int_{t_0}^{t} e^{-(t-s)A} (\partial_s \omega_{0,\theta}(s) + A\omega_{0,\theta}(s) - f(\omega_{0,\theta}(s))) ds \right\|_{L^\infty\left(T_1; H_0^1(\Omega)\right)}$$

into

$$\left\| \int_{t_0}^{t} A^{\frac{1}{2}} e^{-(t-s)A} (f(\omega_{0,\theta}(s)) - g(s)) ds \right\|_{L^\infty(J; L^2(\Omega))} \tag{6}$$

$$+ \left\| \int_{t_0}^{t} A^{\frac{1}{2}} e^{-(t-s)A} (\mathcal{C}_1 \phi_1(s) + \mathcal{C}_0 \phi_0(s)) ds \right\|_{L^\infty(T_1; L^2(\Omega))} \tag{7}$$

Estimation of δ

Since $\hat{u}_{0,\theta}$ and $\hat{u}_{1,\theta} \in V_h \subset L^\infty(\Omega)$, an upper bound of a classical linear interpolation is given for fixed $x \in \Omega$:

$$
\begin{aligned}
|f(\omega_{0,\theta})(t) - g(t)| &\leq \frac{\tau^2}{8} \max_{t \in T_1} \left| \frac{d^2 f(\omega(t))}{dt^2} \right| \\
&= \frac{\tau^2}{8} \max_{t \in T_1} \left| f''[\omega(t)] \left(\frac{d\omega_{0,\theta}}{dt} \right)^2 \right| \\
&= \frac{1}{8} \max_{t \in T_1} |f''[\omega(t)]| \left| (\hat{u}_{1,\theta} - \hat{u}_{0,\theta})^2 \right|,
\end{aligned}
$$

where we denote $f''[\omega(t)]$ be the twice Fréchet derivative of f at $\omega(t)$ for fixed $t \in T_1$

Estimation of δ

It follows

$$\|f(\omega_{0,\theta}) - g(t)\|_{L^2} \leq \frac{M_2}{8}\|f''[\omega_{0,\theta}]\|_{L^\infty(T_1;L^\infty(\Omega))}\|\hat{u}_{1,\theta} - \hat{u}_{0,\theta}\|^2_{H_0^1},$$

where M_2 is a computable constant such that

$$\|u^2\|_{L^2} \leq M_2\|u\|^2_{H_0^1}, \text{ for } u \in H_0^1(\Omega).$$

Let $\lambda_{\min}$ be the minimum eigenvalue of $-\Delta$ and $erf(x) := \frac{2}{\sqrt{\pi}}\int_0^x e^{-x^2}dx$. From the spectrum mapping theorem implies that

$$\int_{t_0}^{t} \|A^{\frac{1}{2}}e^{-(t-s)A}(f(\omega_{0,\theta}(s)) - p(s))\|_{L^2}ds$$

is bounded by

$$\leq \sqrt{\frac{2\pi}{\lambda_{\min}e}}\,\mathrm{erf}\left(\sqrt{\frac{\lambda_{\min}t}{2}}\right)\|f(\omega_{0,\theta}) - g\|_{L^\infty(T_1;L^2(\Omega))}.$$

Estimation of δ

Therefore, we have

$$\left\| \int_{t_0}^{t} A^{\frac{1}{2}} e^{-(t-s)A} (f(\omega_{0,\theta}(s)) - g(s)) ds \right\|_{L^\infty(J;L^2(\Omega))} \leq C_p \alpha^2 \sqrt{\frac{2\pi}{e\lambda_{\min}}} \operatorname{erf}\left(\sqrt{\frac{\lambda_{\min}\tau}{2}} \right),$$

where

$$C_p := \frac{M_2}{8} \|f''[\omega_{0,\theta}]\|_{L^\infty(T_1;L^\infty(\Omega))} \text{ and } \alpha := \|\hat{u}_{1,\theta} - \hat{u}_{0,\theta}\|_{H_0^1}.$$

Estimation of δ

We consider the estimation of

$$\left\|\int_{t_0}^{t} A^{\frac{1}{2}} e^{-(t-s)A}(\mathcal{C}_1\phi_1(s)+\mathcal{C}_0\phi_0(s))ds\right\|_{L^\infty(T_1;L^2(\Omega))}.$$

For $s \in T_1$, since $\phi_0(s)+\phi_1(s)=1$, it sees that

$$\begin{aligned}\mathcal{C}_1\phi_1(s)+\mathcal{C}_0\phi_0(s) &= (\mathcal{C}_1-\mathcal{C}_\theta)\phi_1(s)+(\mathcal{C}_0-\mathcal{C}_\theta)\phi_0(s)+C_\theta \\ &= \mathcal{C}_\theta+\tau^{-1}(1-\theta)(\mathcal{C}_1-\mathcal{C}_0)\,(s-t_0) \\ &\quad +\tau^{-1}\theta(\mathcal{C}_1-\mathcal{C}_0)\,(t_1-s)\,.\end{aligned}$$

Estimation of δ

Therefore, an upper bound of

$$\left\| \int_{t_0}^{t} A^{\frac{1}{2}} e^{-(t-s)A} (\mathcal{C}_1 \phi_1(s) + \mathcal{C}_0 \phi_0(s)) ds \right\|_{L^\infty(T_1; L^2(\Omega))}$$

is given as follows:

$$\begin{aligned} & e^{-\frac{1}{2}} \|\mathcal{C}_\theta\|_{L^2} \int_{t_0}^{t} (t-s)^{-\frac{1}{2}} e^{-(t-s)\frac{\lambda_{\min}}{2}} ds \\ & + d \int_{t_0}^{t} (t-s)^{-\frac{1}{2}} e^{-(t-s)\frac{\lambda_{\min}}{2}} ((1-\theta)\phi_1(s) + \theta\phi_0(s)) ds, \end{aligned}$$

where let $d = e^{-\frac{1}{2}} \|\mathcal{C}_1 - \mathcal{C}_0\|_{L^2}$.

Estimation of δ

Let $r = t - s$. Then, we have

$$\begin{aligned}
& e^{-\frac{1}{2}}\|\mathcal{C}_\theta\|_{L^2}\int_0^{t-t_0} r^{-\frac{1}{2}}e^{-r\frac{\lambda_{\min}}{2}}dr \\
& +\frac{1-\theta}{\tau\sqrt{e}}\|\mathcal{C}_1-\mathcal{C}_0\|_{L^2}\int_0^{t-t_0} r^{-\frac{1}{2}}e^{-r\frac{\lambda_{\min}}{2}}(t-t_0-r)dr \\
& +\frac{\theta}{\tau\sqrt{e}}\|\mathcal{C}_1-\mathcal{C}_0\|_{L^2}\int_0^{t-t_0} r^{-\frac{1}{2}}e^{-r\frac{\lambda_{\min}}{2}}(t_1-t+r)dr \\
= \; & e^{-\frac{1}{2}}\|\mathcal{C}_\theta\|_{L^2}\int_0^{t-t_0} r^{-\frac{1}{2}}e^{-r\frac{\lambda_{\min}}{2}}dr \\
& +\frac{(1-\theta)\phi_1(t)+\theta\phi_0(t)}{\tau\sqrt{e}}\|\mathcal{C}_1-\mathcal{C}_0\|_{L^2}\int_0^{t-t_0} r^{-\frac{1}{2}}e^{-r\frac{\lambda_{\min}}{2}}dr \\
& +\frac{2\theta-1}{\tau\sqrt{e}}\|\mathcal{C}_1-\mathcal{C}_0\|_{L^2}\int_0^{t-t_0} r^{\frac{1}{2}}e^{-r\frac{\lambda_{\min}}{2}}dr.
\end{aligned}$$

Estimation of δ

From

$$\int_0^{t-t_0} r^{\frac{1}{2}} e^{-r\lambda_{\min}/2} dr = -2\sqrt{t-t_0}\lambda_{\min}^{-1} e^{-\lambda_{\min}(t-t_0)/2} + \lambda_{\min}^{-1} \int_0^{t-t_0} r^{-1/2} e^{-r\lambda_{\min}/2} dr,$$

the upper bound is given

$$d_1 \|\mathcal{C}_\theta\|_{L^2} + d_1 \left(\frac{2\theta - 1}{\tau\lambda_{\min}} + ((1-\theta)\phi_1(t) + \theta\phi_0(t)) \right) \|\mathcal{C}_1 - \mathcal{C}_0\|_{L^2} + \frac{2(1-2\theta)}{\tau\sqrt{e}\lambda_{\min}} \|\mathcal{C}_1 - \mathcal{C}_0\|_{L^2} \sqrt{t-t_0} e^{-\lambda_{\min}(t-t_0)/2},$$

where $d_1 = \sqrt{\frac{2\pi}{e\lambda_{\min}}} \operatorname{erf}\left(\sqrt{\frac{\lambda_{\min}(t-t_0)}{2}} \right)$.

Summation of estimation of δ

Let $I(t) : T_1 \to \mathbb{R}$

$$\begin{aligned} I(t) : &= d_1 \|\mathcal{C}_\theta\|_{L^2} \\ &+ d_1 \left(\frac{2\theta - 1}{\tau \lambda_{\min}} + ((1-\theta)\phi_1(t) + \theta\phi_0(t)) \|\mathcal{C}_1 - \mathcal{C}_0\|_{L^2} \right) \\ &+ \frac{2(1-2\theta)}{\tau\sqrt{e}\lambda_{\min}} \|\mathcal{C}_1 - \mathcal{C}_0\|_{L^2} \sqrt{t - t_0} e^{-\lambda_{\min}(t-t_0)/2}, \end{aligned}$$

where $d_1 = \sqrt{\frac{2\pi}{e\lambda_{\min}}} \operatorname{erf}\left(\sqrt{\frac{\lambda_{\min}(t-t_0)}{2}}\right)$. Let $C_p := \frac{M_2}{8} \|f''[\omega_{0,\theta}]\|_{L^\infty(T_1;L^\infty(\Omega))}$ and $\alpha := \|\hat{u}_{1,\theta} - \hat{u}_{0,\theta}\|_{H_0^1}$. δ is bounded as follows:

$$\delta \leq C_p \alpha^2 \sqrt{\frac{2\pi}{e\lambda_{\min}}} \operatorname{erf}\left(\sqrt{\frac{\lambda_{\min}\tau}{2}}\right) + \max_{t \in T_1} I(t).$$

3. Numerical example

Numerical example

Let Ω be an unit square domain. We consider existence and uniqueness of the solution in the following heat equations:

$$\begin{cases} \partial_t u - \Delta u = u^2 & \text{in } (0,\infty)\times\Omega, \\ u|_{\partial\Omega} = 0 & \text{on } (0,\infty)\times\partial\Omega, \\ u(0,x) = \sin(\pi x)\sin(\pi y) & \text{in } \Omega. \end{cases} \tag{8}$$

to consider which scheme gives a shaper absolute error estimate ($\theta = 0, 1$ and $1/2$).

Setting for numerical computation

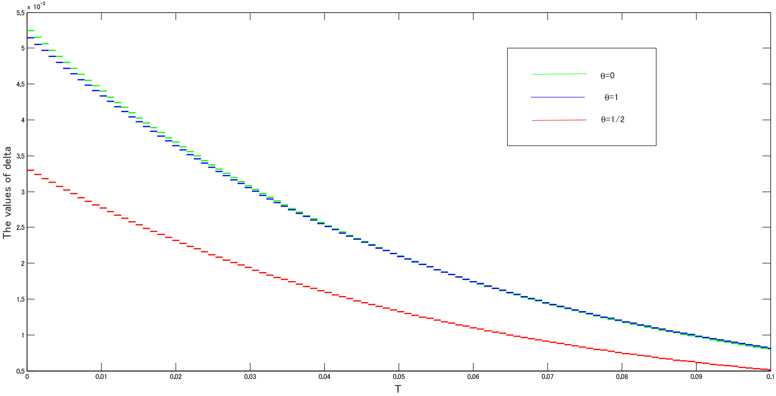
- Windows 7 Professional, Intel(R) Core(TM) i7 CPU 860 2.80GHz
- 16 GB Memory
- MATLAB 2012a (toolbox INTLAB ver7 [1])
- For $N \in \mathbb{N}$, let a finite dimensional space $V_N \subset \mathcal{D}(A)$ by

$$V_N = \left\{ u_h = \sum_{k,l=1}^{N} a_{k,l} \sin(k\pi x) \sin(l\pi y) \ : \ a_{k,l} \in \mathbb{R} \right\}.$$

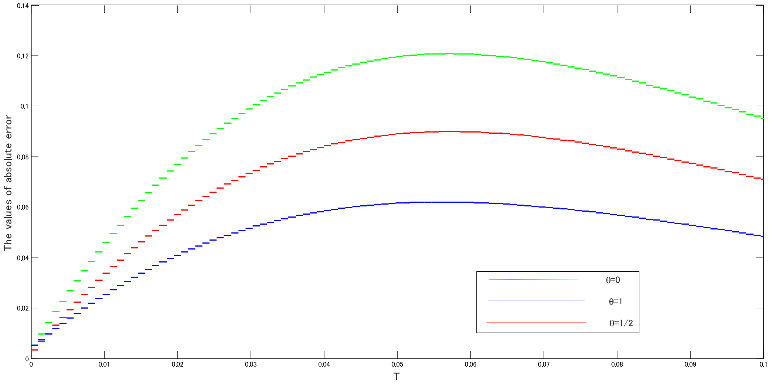
- We set $N = 7$ and $\tau = 2^{-10}$ equally.

[1] S.M. Rump. INTLAB - INTerval LABoratory. In Tibor Csendes, editor, Developments in Reliable Computing, pages 77-104. Kluwer Academic Publishers, Dordrecht, 1999.

Verification of δ



Verification of $\|u-\omega\|_{L^\infty(T_k;H_0^1)}$

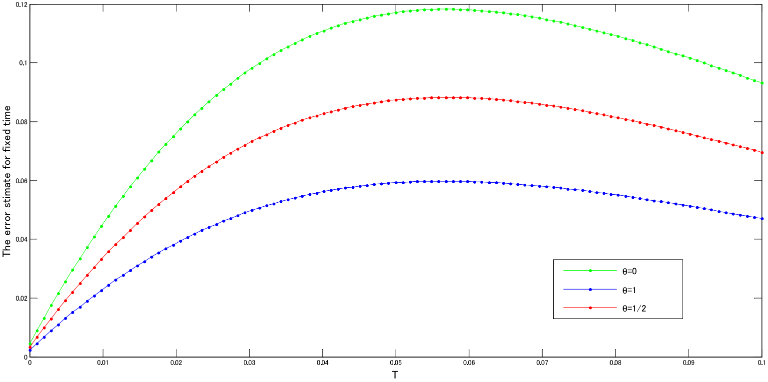


Essential estimate to obtain absolute error

To get rigorous estimate absolute error $\|\cdot\|_{L^\infty(T_1;H_0^1(\Omega))}$, we need to obtain sharp estimates:

$$\begin{cases} \delta := \left\| \int_{t_0}^{t} e^{-(t-s)A}(\partial_s \omega_0(s) + A\omega_0(s) - f(\omega_0(s)))ds \right\|_{L^\infty\left(T_1;H_0^1(\Omega)\right)}, \\ \varepsilon := \|u(t_0) - \hat{u}_0\|_{H_0^1}. \end{cases}$$

Verification of $\|u(t_k) - \hat{u}_k\|_{H_0^1}$



Conclusion

- The scheme with $\theta = 1/2$ gives a sharper estimate of δ . (Crank-Nicolson scheme)
- The scheme with $\theta = 1$ gives a sharper estimate of the absolute error: $\|u - \omega\|_{L^\infty(T_1;H_0^1(\Omega))}$ (Backward Euler scheme)
- The reason is that the value of the absolute error is dependent not only δ but also ε

$$\Downarrow$$

The scheme with $\theta = 1$ has an effective estimate in this numerical example.
(Backward Euler scheme)