A method of verified computations for nonlinear parabolic equations

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Joint with

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SCAN2014@Würzburg

September 23, 2014

Nonlinear parabolic equations

Let Ω be a bounded polygonal domain in $\mathbb{R}^2$.

$$(P)\ \begin{cases} \partial_t u + Au = f(u) & \text{in } (0,\infty)\times\Omega, \\ u(t,x) = 0 & \text{on } (0,\infty)\times\partial\Omega, \\ u(0,x) = u_0(x) & \text{in } \Omega, \end{cases}$$

- $\partial_t u = \frac{du}{dt}$,
- f : a twice Fréchet differentiable nonlinear mapping,
- $u_0 \in H_0^1(\Omega)$ is a given initial function,
- $A : D(A) \subset H_0^1(\Omega) \to L^2(\Omega)$, self-adjoint.

$L^p(\Omega)$: the set of L^p-functions,

$H^1(\Omega)$: the first order Sobolev space of $L^2(\Omega)$,

$H_0^1(\Omega) := \{u \in H^1(\Omega) : u = 0 \text{ on } \partial\Omega\}$.

Differential operator A

We set

$$A := -\sum_{1\le i,j\le 2} \frac{\partial}{\partial x_j}\left(a_{ij}(x)\frac{\partial}{\partial x_i}\right),$$

where $a_{ij}(x)(= a_{ji}(x))$ is in $W^{1,\infty}(\Omega)$ and satisfies

$$\sum_{1\le i,j\le 2} a_{ij}(x)\xi_i\xi_j \ge \mu|\xi|^2,\ \forall x\in\Omega,\ \forall \xi\in\mathbb{R}^2 \text{ with } \mu>0.$$

$a: H_0^1(\Omega)\times H_0^1(\Omega)\to\mathbb{R}$ is a bilinear form:

$$a(u,v) := \sum_{1\le i,j\le 2}\left(a_{ij}(x)\frac{\partial u}{\partial x_i},\frac{\partial v}{\partial x_j}\right)_{L^2}$$

satisfying

$$a(u,u)\ge \mu\|u\|_{H_0^1}^2,\ |a(u,v)|\le M\|u\|_{H_0^1}\|v\|_{H_0^1},\ \forall u,v\in H_0^1(\Omega).$$

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$$a(u,u)\ge\mu\|u\|^2_{H_0^1},\ |a(u,v)|\le M\|u\|_{H_0^1}\|v\|_{H_0^1},\ \forall u,v\in H_0^1(\Omega).$$

Weak solutions (aim of this study)

The weak form of (P):

find $u(t) := u(t,\cdot) \in H_0^1(\Omega)$ for $t > 0$ s.t.

$$(\partial_t u(t), v)_{L^2} + a(u(t), v) = (f(u(t)), v)_{L^2},\ \forall v \in H_0^1(\Omega).$$

Let $T := (0, t']$ for a fixed $0 < t' < \infty$.
We seek a weak solution in the Banach space $L^\infty\left(T; H_0^1(\Omega)\right)$ with the norm:

$$\|u\|_{L^\infty\left(T; H_0^1(\Omega)\right)} := \operatorname*{ess\,sup}_{t \in T} \|u(t)\|_{H_0^1}.$$

Then, we enclose the weak solution in a neighborhood of a numerical solution.

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Previous studies

[1] M.T. Nakao, T. Kinoshita and T. Kimura,

"*On a posteriori estimates of inverse operators for linear parabolic initial-boundary value problems*", Computing 94(2-4), 151–162, 2012.

[2] M.T. Nakao, T. Kimura and T. Kinoshita,

"*Constructive A Priori Error Estimates for a Full Discrete Approximation of the Heat Equation*", Siam J. Numer. Anal., 51(3), 1525–1541, 2013.

[3] T. Kinoshita, T. Kimura and M.T. Nakao,

"*On the a posteriori estimates for inverse operators of linear parabolic equations with applications to the numerical enclosure of solutions for nonlinear problems*", Numer. Math, 126(4), 679–701, 2014. Online First, 2013.

Features & Comparison

- The enclosure of weak solutions is a ball of

$$L^2\left(J; H_0^1(\Omega)\right) := \left\{ u(t) \in H_0^1(\Omega) : \int_J \|u(t)\|_{H_0^1}^2 dt < \infty \right\}.$$

- Use the full discretization projection P_h^k cleverly;

$$\left\|u - P_h^k u\right\|_{L^2\left(J; H_0^1(\Omega)\right)} \le C(h,k) \left\|(\partial_t - \Delta) u\right\|_{L^2\left(J; L^2(\Omega)\right)}$$

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for $\forall u \in H^1\left(J; L^2(\Omega)\right) \cap L^2\left(J; D(\Delta)\right)$.

- Analitic semigroup $e^{-t\mathcal{A}}$ generated by $-\mathcal{A}$[1].

[1] $\mathcal{A} : H_0^1(\Omega) \to H^{-1}(\Omega)$ s.t. $\langle \mathcal{A}u, v\rangle := a(u,v),\ \forall v \in H_0^1(\Omega)$.

Semigroup theory

It is fact that the weak form of A, which is denoted by $-\mathcal{A}$, generates the analytic semigroup $\{e^{-t\mathcal{A}}\}_{t\geq 0}$ over $H^{-1}(\Omega)$. The following abstract problem has an unique solution:

$$\partial_t u + \mathcal{A}u = 0,\ u(0,x) = u_0 \Longrightarrow \exists u = e^{-t\mathcal{A}}u_0.$$

Fact

Let $x \in D(\mathcal{A})$ and λ_0 be a positive number. $\mathcal{A}$ satisfies

$$\langle -\mathcal{A}x, x\rangle \leq 0, \quad R(\lambda_0 I + \mathcal{A}) = H^{-1}(\Omega).$$

Then, there exists an analytic semigroup $\{e^{-t\mathcal{A}}\}_{t\geq 0}$ generated by $-\mathcal{A}$.

Proofs are found in several textbooks.

Two-steps procedure

1. Check a sufficient condition for proving existence and local uniqueness of the weak solution in a time interval $J := (t_0, t_1]$ $(0 \le t_0 < t_1 < \infty)$.
   If it succeeds, then the weak solution is in the ball[2]

$$B(\omega, \rho) := \left\{ y \in L^\infty\left(J; H_0^1(\Omega)\right) : \|y - \omega\|_{L^\infty\left(J; H_0^1(\Omega)\right)} \le \rho \right\}.$$

2. Connect each enclosure of the weak solution by iterating numerical verification. Extend a time interval in which existence of the weak solution is guaranteed.
   We call this 2nd step *"concatenation scheme"*.

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1. Verification theorem

Considered problem

$J = (t_0, t_1]$: arbitrary time interval. $\tau := t_1 - t_0$.

$$(P_J) \begin{cases} \partial_t u + Au = f(u) & \text{in } J \times \Omega, \\ u(t,x) = 0 & \text{on } J \times \partial\Omega, \\ u(t_0, x) = \zeta(x) & \text{in } \Omega, \end{cases}$$

where ζ is an initial function in $H_0^1(\Omega)$.

$V_h \subset H_0^1(\Omega)$: a finite dimensional subspace.
Starts from: $\hat{u}_0, \hat{u}_1 \in V_h$

$$\omega(t) = \hat{u}_0 \phi_0(t) + \hat{u}_1 \phi_1(t), \ t \in J,$$

where $\phi_k(t)$ is a linear Lagrange basis: $\phi_k(t_j) = \delta_{kj}$ (δ_{kj} is a Kronecker's delta).

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Theorem (Verification theorem)

Suppose that ζ is in the ε_0-neighborhood of $\hat{u}_0$; δ satisfies

$$\left\| \int_{t_0}^{t} e^{-(t-s)\mathcal{A}} (\partial_t \omega(s) + \mathcal{A}\omega(s) - f(\omega(s))) ds \right\|_{L^\infty\left(J;H_0^1(\Omega)\right)} \le \delta$$

and a local Lipschitz bound $L_{\rho_0} > 0$ for $\rho \ge \rho_0 > 0$ satisfies

$$\|f(\varphi) - f(\psi)\|_{L^\infty(J;L^2(\Omega))} \le L_{\rho_0} \|\varphi - \psi\|_{L^\infty\left(J;H_0^1(\Omega)\right)},$$

where $\forall \varphi, \psi \in B(\omega, \rho)$. If there exists $\rho > 0$ such that

$$\frac{M}{\mu}\varepsilon_0 + \frac{2}{\mu}\sqrt{\frac{M\tau}{e}} L_\rho \rho + \delta < \rho,$$

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Sketch of proof

Let $z(t) \in H_0^1(\Omega)$ for a fixed $t \in J$. We put $u(t) = \omega(t) + z(t)$. For any $v \in H_0^1(\Omega)$,

$$
\begin{aligned}
& (\partial_t z(t), v)_{L^2} + a(z(t), v) \\
= \; & (f(u(t)), v)_{L^2} - ((\partial_t \omega(t), v)_{L^2} + \langle \mathcal{A}\omega(t), v \rangle) \\
= \; & \langle g(z(t)), v \rangle ,
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where $g(z(t)) = f(u(t)) - (\partial_t \omega(t) + \mathcal{A}\omega(t)) \in H^{-1}(\Omega)$.

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where $g(z(t)) = f(u(t)) - (\partial_t \omega(t) + \mathcal{A}\omega(t)) \in H^{-1}(\Omega)$.

Define $S : L^\infty(J; H_0^1(\Omega)) \to L^\infty(J; H_0^1(\Omega))$ with the analytic semigroup $e^{-t\mathcal{A}}$ by

$$S(z) := e^{-(t-t_0)\mathcal{A}}(\zeta - \hat{u}_0) + \int_{t_0}^{t} e^{-(t-s)\mathcal{A}} g(z(s)) ds.$$

Sketch of proof

For $\rho > 0$, let $Z := \left\{ z \in L^\infty\left(J; H_0^1(\Omega)\right) : \|z\|_{L^\infty\left(J; H_0^1(\Omega)\right)} \le \rho \right\}$.
On the basis of Banach's fixed-point theorem, we show a sufficient condition of S having a fixed-point in Z.

Then it follows

$$\|S(z)\|_{L^\infty\left(J; H_0^1(\Omega)\right)} \le \frac{M}{\mu}\varepsilon_0 + \frac{2}{\mu}\sqrt{\frac{M\tau}{e}} L_\rho \rho + \delta.$$

$\|S(z)\|_{L^\infty\left(J; H_0^1(\Omega)\right)} \le \rho$ holds from the condition of the theorem. It implies that $S(z) \in Z$.

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$\|S(z)\|_{L^\infty\left(J; H_0^1(\Omega)\right)} < \rho$ holds from the condition of the theorem. It implies that $S(z) \in Z$.

Sketch of proof

If z_1, z_2 in Z,

$$\|S(z_1) - S(z_2)\|_{L^\infty\left(J;H_0^1(\Omega)\right)} \leq \frac{2}{\mu}\sqrt{\frac{M\tau}{e}}L\left(\rho\right)\|z_1 - z_2\|_{L^\infty\left(T_k;H_0^1(\Omega)\right)}$$

holds. The condition of theorem also implies

$$\frac{2}{\mu}\sqrt{\frac{M\tau}{e}}L\left(\rho\right) < 1.$$

Therefore, S becomes a contraction mapping.
Banach's fixed-point theorem yields that there uniquely exists a fixed-point of S in Z.

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Theorem (A posteriori error estimate)

Suppose that the weak solution $u(t)$, $t \in J$, of (P_J) is in $B(\omega, \rho)$. Assuming that $\tilde{\delta} > 0$ satisfies

$$\left\| \int_{t_0}^{t_1} e^{-(t_1 - s)\mathcal{A}} \left(\partial_t \omega(s) + \mathcal{A}\omega(s) - f(\omega(s))\right) ds \right\|_{H_0^1} \le \tilde{\delta}.$$

Then, the following a posteriori error estimate holds:

$$\|u(t_1) - \hat{u}_1\|_{H_0^1} \le \frac{M}{\mu} e^{-\tau\lambda_{\min}} \varepsilon_0 + \frac{2}{\mu}\sqrt{\frac{M\tau}{e}} L_\rho \rho + \tilde{\delta} =: \varepsilon_1.$$

2. Concatenation scheme

Considered problem

Let $T = (0, t']$ for $t' > 0$. We divide T into n intervals by $0 = t_0 < t_1 < \cdots < t_n = t'$ for $n \in \mathbb{N}$.
Define $T_k := (t_{k-1}, t_k]$ and $\tau_k := t_k - t_{k-1}$ $(k = 1, 2, ..., n)$.

$$(P_T)\begin{cases} \partial_t u + Au = f(u) & \text{in } T \times \Omega, \\ u(t,x) = 0 & \text{on } T \times \partial\Omega, \\ u(t_0, x) = u_0(x) & \text{in } \Omega, \end{cases}$$

where $u_0 \in L^2(\Omega)$ is a given initial function.

From $\hat{u}_k \in V_h$, we construct a numerical solution $\omega(t)$ by

$$\omega(t) := \sum_{k=0}^{n} \hat{u}_k \phi_k(t), \ \ t \in T,$$

where $\phi_k(t)$ is a piecewise linear Lagrange basis.

Considered problem

Let $T = (0, t']$ for $t' > 0$. We divide T into n intervals by $0 = t_0 < t_1 < \cdots < t_n = t'$ for $n \in \mathbb{N}$.
Define $T_k := (t_{k-1}, t_k]$ and $\tau_k := t_k - t_{k-1}$ $(k = 1, 2, ..., n)$.

$$(P_T) \begin{cases} \partial_t u + Au = f(u) & \text{in } T \times \Omega, \\ u(t,x) = 0 & \text{on } T \times \partial\Omega, \\ u(t_0, x) = u_0(x) & \text{in } \Omega, \end{cases}$$

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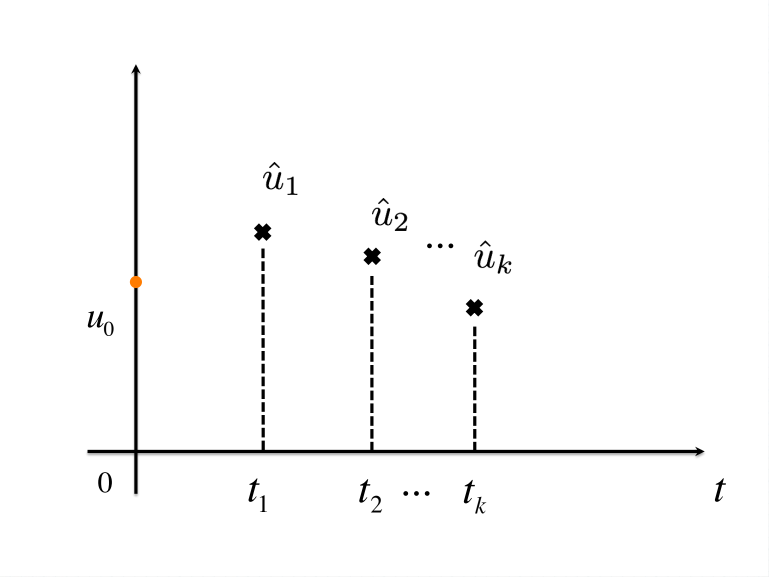
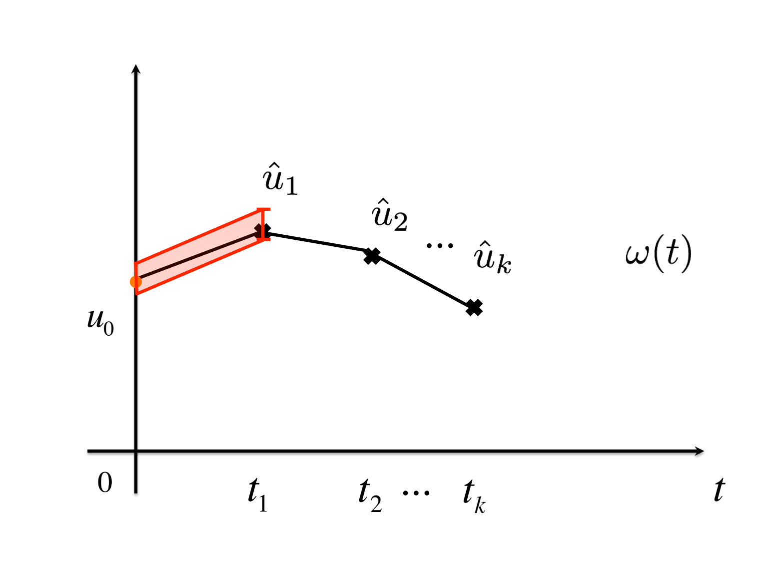
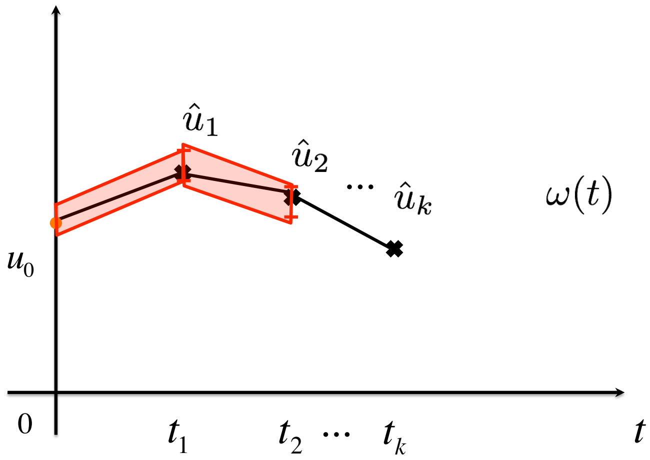
Concatenation scheme



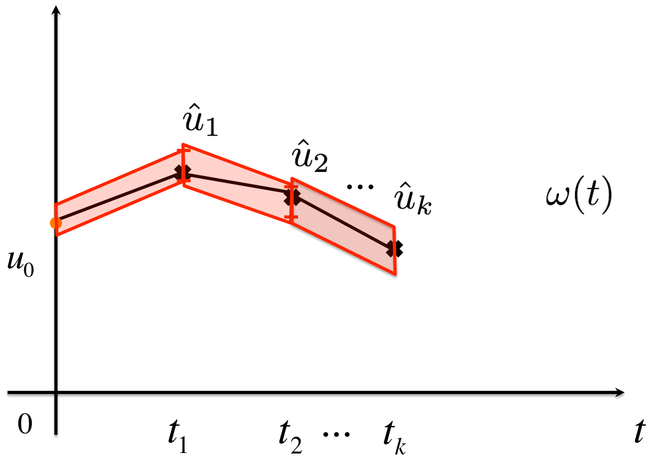
Concatenation scheme



Concatenation scheme



Concatenation scheme



Computational example

Fujita-type parabolic problem

Let $\Omega := (0,1)^2$ be an unit square domain in $\mathbb{R}^2$.

$$(F)\begin{cases} \partial_t u - \Delta u = u^2 & \text{in } (0,\infty)\times\Omega, \\ u(t,x) = 0 & \text{on } (0,\infty)\times\partial\Omega, \\ u(0,x) = u_0(x) & \text{in } \Omega, \end{cases}$$

Let $\gamma > 0$ be an parameter of the initial function:

$$u_0(x) = \gamma x_1(1-x_1)x_2(1-x_2).$$

V_h: the quadratic conforming finite elements (P_2-elements),
h: a spatial mesh size.

The backward Euler method is employed.
τ: time step.

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Computational results

Table: $\|u(t_k) - \hat{u}_k\|_{H_0^1} \leq \varepsilon_k$, $\|u - \omega\|_{L^\infty\left(T_k; H_0^1(\Omega)\right)} \leq \rho_k$ when $h = 2^{-4}$, $\tau = 2^{-8}$, and $\gamma = 1$.

$T_k = (t_{k-1}, t_k]$	ε_k	ρ_k
(0,0.0039062]	0.020155	0.037646
(0.0039062,0.0078125]	0.030051	0.041554
(0.0078125,0.011719]	0.038089	0.049313
(0.011719,0.015625]	0.044657	0.055699
(0.015625,0.019531]	0.050001	0.060873
⋮	⋮	⋮
(0.48047,0.48438]	0.00013041	0.00014184
(0.48438,0.48828]	0.00012186	0.00013255
(0.48828,0.49219]	0.00011388	0.00012386
(0.49219,0.49609]	0.00010641	0.00011573
(0.49609,0.5]	9.9431E-5	0.00010813

Computational results

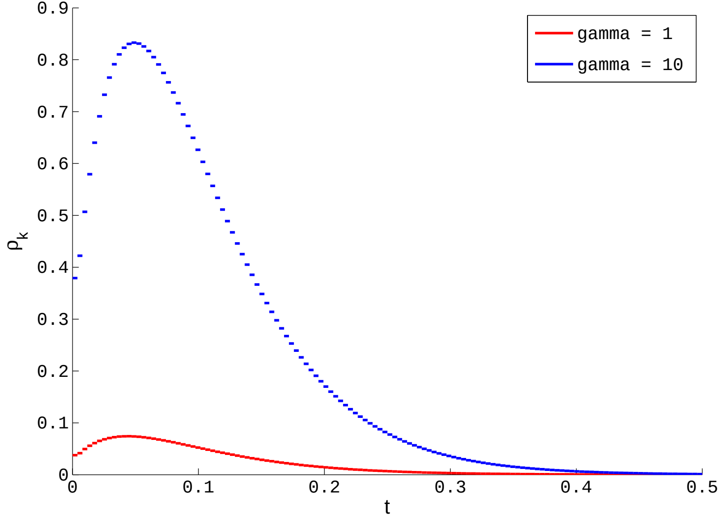

Figure: $h = 2^{-4}$, $\tau = 2^{-8}$, $\gamma = 1,\ 10$.

Computational results

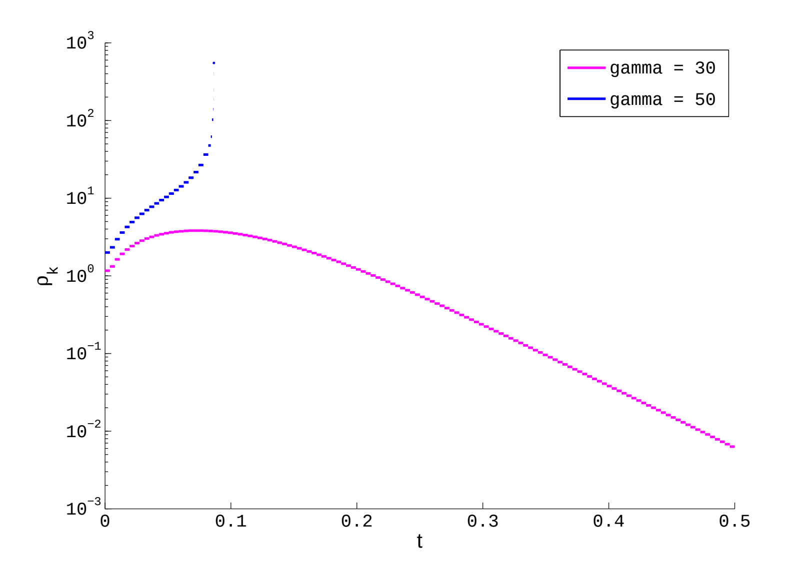

Figure: $h = 2^{-4}$, $\tau = 2^{-8}$, $\gamma = 30,\ 50.$

Conclusion

- Enclose the weak solution of parabolic problems
- Verification theorem
- Concatenation scheme of a verified inclusion

Further works

- Shape error estimate is desired.
- How about strong solution?

$\longrightarrow$ the next talk...

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Thank you for kind attention!