Optimal Preconditioning for the Interval Parametric Gauss–Seidel Method

Milan Hladík

Faculty of Mathematics and Physics,
Charles University in Prague, Czech Republic
http://kam.mff.cuni.cz/~hladik/

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Interval Linear Equations

$$Ax = b, \quad A \in \mathbf{A},\ b \in \mathbf{b},$$

where

$$\mathbf{A} := [\underline{A}, \overline{A}] = [A^c - A^\Delta, A^c + A^\Delta],$$
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The Solution Set

$$\Sigma := \{x \in \mathbb{R}^n : \exists A \in \mathbf{A}, \exists b \in \mathbf{b} : Ax = b\}.$$

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Problem formulation

Find a tight interval vector enclosing Σ.

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Let $C \in \mathbb{R}^{n\times n}$. Preconditioning is a relaxation to

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The Interval Gauss–Seidel Method

$$\mathbf{z}_i := \frac{1}{(C\mathbf{A})_{ii}}\left((C\mathbf{b})_i - \sum_{j\neq i}(C\mathbf{A})_{ij}\mathbf{x}_j\right), \quad \mathbf{x}_i := \mathbf{x}_i \cap \mathbf{z}_i,$$

for $i = 1, \dots, n$.

Interval Parametric Systems

Interval Parametric System

$$A(p)x = b(p), \quad p \in \mathbf{p},$$

where

$$A(p) = \sum_{k=1}^{K} A^k p_k, \quad b(p) = \sum_{k=1}^{K} b^k p_k,$$

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Preconditioning and Relaxation

Relaxation to $Ax = b$, where

$$A \in \mathbf{A} := \sum_{k=1}^{K} (CA^k)\mathbf{p}_k, \quad b \in \mathbf{b} := \sum_{k=1}^{K} (Cb^k)\mathbf{p}_k.$$

The Parametric Interval Gauss–Seidel Method

$$\mathbf{z}_i := \frac{1}{\left(\sum_{k=1}^{K}(CA^k)_{ii}\mathbf{p}_k\right)} \left(\sum_{k=1}^{K}(Cb^k)_i\mathbf{p}_k - \sum_{j \neq i} \left(\sum_{k=1}^{K}(CA^k)_{ij}\mathbf{p}_k \right) \mathbf{x}_j \right),$$

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Optimal Preconditioner

Various criteria of optimality:

- minimize the resulting width, that is, the objective is $\min 2z_i^{\Delta}$,
- minimize the resulting upper bound, that is, the objective is $\min \overline{z}_i$,
- maximize the resulting lower bound, that is, the objective is $\max \underline{z}_i$.

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Minimal Width Preconditioner

Preliminaries

- For simplicity assume that $0 \in \mathbf{x}$ and $0 \in \mathbf{z}$
- Denote by c the ith row of C,
- Normalize c such that the denominator has the form of $[1, r]$ for some $r \geq 1$.

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Interval Gauss–Seidel Step

Then the operation of the ith step of the Interval Gauss–Seidel iteration is simplified to

$$\mathbf{z}_i := \sum_{k=1}^{K} (cb^k)\mathbf{p}_k - \sum_{j \neq i} \left(\sum_{k=1}^{K} (cA^k_{*j})\mathbf{p}_k \right) \mathbf{x}_j$$

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The objective is $\min z_i^{\Delta}$.

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$$\text{minimize the width of} \quad \sum_{k=1}^{K} (cb^k)\mathbf{p}_k - \sum_{j \neq i} \left(\sum_{k=1}^{K} (cA_{*j}^k)\mathbf{p}_k \right) \mathbf{x}_j.$$

Denote

$$\begin{aligned}
\beta_k &:= |cb^k|, \quad k = 1, \ldots, K, \\
\alpha_{jk} &:= |cA_{*j}^k|, \quad j = 1, \ldots, n, \ k = 1, \ldots, K, \\
\eta_j &:= \overline{\left(\textstyle\sum_{k=1}^{K} (cA_{*j}^k)\mathbf{p}_k \right) \mathbf{x}_j}, \quad j \neq i, \\
\psi_j &:= \underline{\left(\textstyle\sum_{k=1}^{K} (cA_{*j}^k)\mathbf{p}_k \right) \mathbf{x}_j}, \quad j \neq i.
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$$\min \sum_{k=1}^{K} 2p_k^{\Delta}\beta_k + \sum_{j\neq i}(\eta_j - \psi_j),$$

subject to

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$$\eta_j \geq c\sum_{k=1}^{K} A_{*j}^k p_k^c \underline{x}_j \pm \sum_{k=1}^{K} p_k^{\Delta} \underline{x}_j \alpha_{jk}, \quad j \neq i,$$

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\psi_j &\leq c\textstyle\sum_{k=1}^{K} A^k_{*j} p^c_k \overline{x}_j - \sum_{k=1}^{K} p^{\Delta}_k \overline{x}_j \alpha_{jk}, \quad j \neq i,
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and the condition that the denominator is has the form of $[1, r]$

$$c\sum_{k=1}^{K} A_{*i}^k p_k^c - \sum_{k=1}^{K} p_k^{\Delta}\alpha_{ik} = 1.$$

Minimal Width Preconditioner

Optimization problem.

- Optimal preconditioner C found by n linear programming (LP) problems.
- each LP has $Kn + K + 3n - 2$ unknowns c, β_k, α_{jk}, η_j, and ψ_j, and $2Kn + 4n - 3$ constraints
- C needn't be calculated in a verified way.
- The problem is effectively solved in polynomial time.

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Practical Implementation

- Call the standard version using midpoint inverse preconditioner (or any other method),
- and after that tighten the enclosure by using an optimal preconditioner C.
- In our examples: one iteration with minimization of the upper bound, and one with maximization of the upper bound.

Example I

Example (Popova, 2002)

$$A(p) = \begin{pmatrix} 1 & p_1 \\ p_1 & p_2 \end{pmatrix}, \quad b(p) = \begin{pmatrix} p_3 \\ p_3 \end{pmatrix}, \quad p \in \mathbf{p} = ([0,1], -[1,4], [0,2])^T.$$

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Initial enclosure by the Parametric Interval Gauss–Seidel Method with midpoint inverse preconditioner:

- direct version: 7.66% of the width on average reduced
- residual form: 0% of the width on average reduced

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- direct version: 7.66% of the width on average reduced
- residual form: 0% of the width on average reduced

Initial enclosure as the interval hull of the relaxed system:

- direct version: 50% of the width on average reduced
- residual form: 12.56% of the width on average reduced

Example II

Example (Popova and Krämer, 2008)

$$A(p) = \begin{pmatrix} 30 & -10 & -10 & -10 & 0 \\ -10 & 10 + p_1 + p_2 & -p_1 & 0 & 0 \\ -10 & -p_1 & 15 + p_1 + p_3 & -5 & 0 \\ -10 & 0 & -5 & 15 + p_4 & 0 \\ 0 & 0 & -5 & 5 & 1 \end{pmatrix}, \quad b(p) = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix},$$

where $p \in \mathbf{p} = [8, 12] \times [4, 8] \times [8, 12] \times [8, 12]$.

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Initial enclosure by the Parametric Interval Gauss–Seidel Method with midpoint inverse preconditioner:

- direct version: 15% of the width on average reduced
- residual form: 0% of the width on average reduced

Conclusion

Summary

- Optimal preconditioning matrix for the parametric interval Gauss–Seidel iterations.
- It can be computed effectively by linear programming.
- Preliminary results show that sometimes can reduce overestimation of the standard enclosures.

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Directions for Further Research

- Other types of optimality of preconditioners (S-preconditioners, pivoting preconditioners, etc.)
- Optimal preconditioners for other methods than the parametric interval Gauss–Seidel one.

References

M. Hladík.
Enclosures for the solution set of parametric interval linear systems.
Int. J. Appl. Math. Comput. Sci., 22(3):561–574, 2012.

R. B. Kearfott.
Preconditioners for the interval Gauss–Seidel method.
SIAM J. Numer. Anal., 27(3):804–822, 1990.

R. B. Kearfott, C. Hu, and M. Novoa III.
A review of preconditioners for the interval Gauss–Seidel method.
Interval Comput., 1991(1):59–85, 1991.

A. Neumaier.
New techniques for the analysis of linear interval equations.
Linear Algebra Appl., 58:273–325, 1984.

E. Popova.
Quality of the solution sets of parameter-dependent interval linear systems.
ZAMM, Z. Angew. Math. Mech., 82(10):723–727, 2002.

E. D. Popova and W. Krämer.
Visualizing parametric solution sets.
BIT, 48(1):95–115, 2008.


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