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Non-arithmetic approach to dealing with uncertainty in fuzzy arithmetic

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Fuzzy Numbers

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Fuzzy logic was introduced mainly due to the fact that people do not communicate or even think with some precise numbers or categories.

Fuzzy numbers are mainly used to refer to some experts (or other uncertain) evaluation of some measurement of an object or event.
It can be expert's view in economics, engineering, etc., or an uncertain number obtained during machine learning procedure.

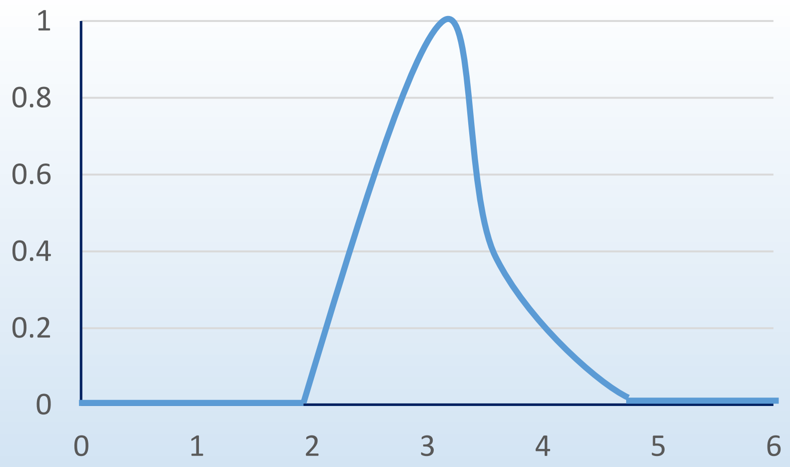
For example, let's consider the maximum amount you can pay for a notebook. Suppose, your upper price border is 1000 euro. But is it really the upper bound? What about 1001 euro? 1010 euro? 1050 euro? No one can say the number which is the exact border.

And we represent fuzzy numbers with some quantity, where the top is the number you are completely sure about, and then we have some kind of slope on the both sides.

There are several ways to represent Fuzzy numbers:

- L-R functions (for left and right slope)
- For a chosen Δx, we work with the f(x) for each steps
- Or we use interval arithmetic (the most popular one)

The fuzzy number can have any shape, as far is it's left and right sides are monotonic (non-negative and non-positive respectively).



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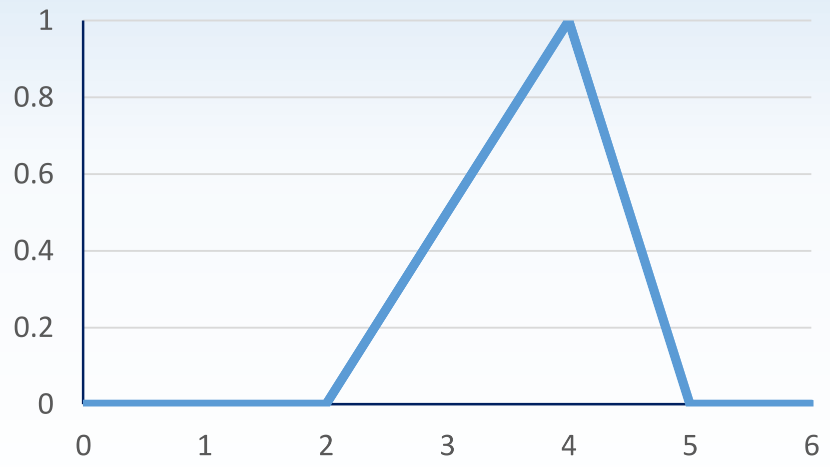
Interval Arithmetic is one of the most common ways to deal with fuzzy numbers.

In this case each fuzzy number is represented as a set of intervals for each α-level, where α -levels are chosen based on required precision.



For example, let's consider the triangular-shaped number with the top on 4, and bottoms on 2 and 5.

For simplicity, I will use only bottom level (α = 0),
but the same idea can be use for any α-level directly.

Let's represent [a, b] as the border for chosen α-level. In the example above for α = 0 it's [2, 5], for α = 1 it's [4, 4].
To refer to the fuzzy number itself I will use capital letters, like **X** = [a, b].

Such approach allows us to use all interval arithmetic tools, including basic arithmetic operations:

- Addition: [a, b] + [c, d] = [a + c, b + d]
- Subtraction: [a, b] – [c, d] = [a – d, b – c]
- Multiplication: [a, b] · [c, d] = [min(ac, ad, bc, bd), max(ac, ad, bc, bd)]
- Division: [a, b] / [c, d] = [min(a/c, a/d, b/c, b/d), max(a/c, a/d, b/c, b/d)], when 0 is not in [b, c]

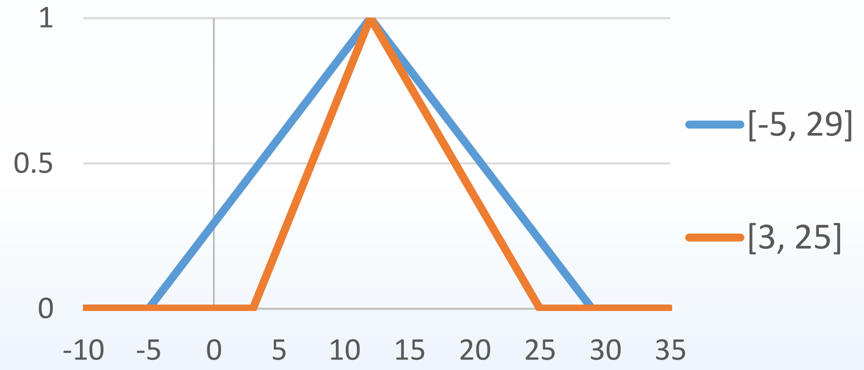
Problem

The main problem is: fuzzy numbers in interval arithmetic form neither field, ring nor even a group. Monoid is as far as it goes.

The addition and multiplication operations are associative and even commutative…
But there still are problems:

1) No inverse element for addition or multiplication:
   - **X** – **X** ≠ 0 and vary for different **X**: [a, b] – [a, b] = [a – b, b – a].
   - **X** / **X** ≠ 1 and also vary: [a, b] / [a, b] = [min(a/a, a/b, b/a, b/b), max(a/a, a/b, b/a, b/b)].

2) Not distributive: **X** · **B** + **X** · **C** ≠ **X** · (**B** + **C**)
   For example:
   - [3, 5] · [5, 7] – [3, 5] · [2, 4] = [15, 35] – [6, 20] = [-5, 29]
   - [3, 5] · ([5, 7] – [2, 4]) = [3, 5] · [1, 5] = [3, 25]



3) As follows from the previous point – polynomials are even of a greater problem, as:
   - (**X** + **A**) · **X** ≠ $\mathbf{X}^2$ + **A** · **X**, not to mention polynomials of higher degrees.

All these problems produce a lot of uncertainty when trying to solve equations with fuzzy numbers.

Mathematical approach and its problem

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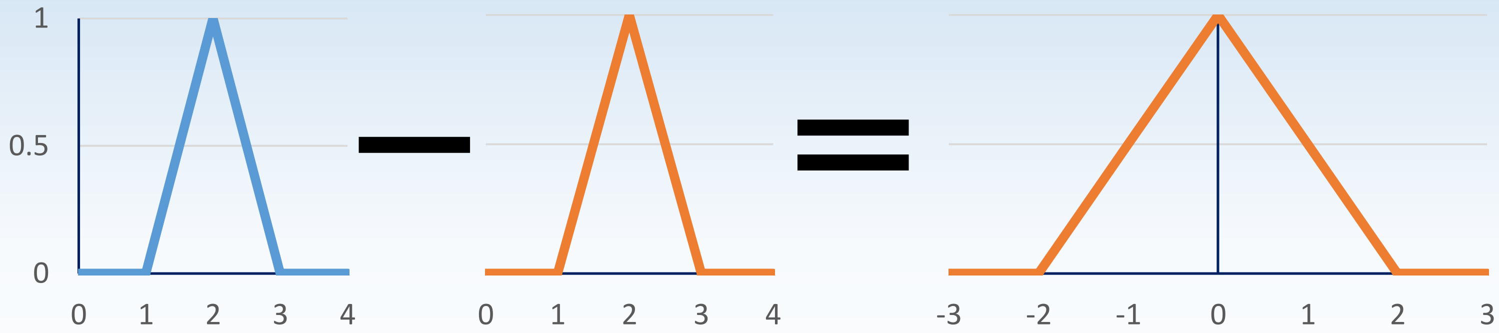
Some researchers try to produce an alternative mathematical approach, to solve these problems. But I consider, that the idea of finding such approach is ideologically wrong. Let's consider some examples:

Let's consider the simplest case:
X – **X** ≠ **0** (or [0, 0])

But why it should be equal to zero?



You can make "–" to be an operation, such that **X** – **X** = **0**. And we can even ignore the fact that such operation will have many negative sides (limitation for usage, inapplicability of classical interval arithmetic, etc.).

But the main problem is: should **X** – **X** be really equal to zero? If **X** is just an abstract fuzzy number – my answer is "no, it shouldn't". Just as in statistics, the elder brother of fuzzy theory, subtraction should increase variance.

The same point is for non-distributivity: **X** · (**B** + **C**) ≠ **X** · **B** + **X** · **C**. This inequality also shouldn't be transferred to equality.

And the reason for both cases is: it would ruin the intuition of fuzzy arithmetic.
Instead I propose easy non-arithmetic approach to reduce this uncertainty, applicable to many cases.

Non-arithmetic approach

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The idea is quite simple: before using interval arithmetic, it is necessary to make meaningful analysis of each fuzzy number or variable. A sort of “common sense” analysis.

Let’s consider cases above:

1) $\mathbf{X_1} - \mathbf{X_2} \neq \mathbf{0}$, where $\mathbf{X_1} = \mathbf{X_2}$ as fuzzy numbers:

$\mathbf{X_1}$ = [a, b], $\mathbf{X_2}$ = [a, b], where $\mathbf{X_1} = \mathbf{X_2}$, with completely equal distributions on each α-level.

- If $\mathbf{X_1}$ and $\mathbf{X_2}$ are completely independent variables, then we should leave It be, and use $\mathbf{X_1} - \mathbf{X_2}$ = [a – b, b – a].
- But if $\mathbf{X_1}$ and $\mathbf{X_2}$ reflect the same measurement of the same object or event, then we should use $\mathbf{X_1} - \mathbf{X_2} = \mathbf{0}$.

Same approached should be used with $\mathbf{X_1} / \mathbf{X_2} = 1$.

2) $\mathbf{X_1} \cdot \mathbf{B} + \mathbf{X_2} \cdot \mathbf{C} \neq \mathbf{X} \cdot (\mathbf{B} + \mathbf{C})$, where $\mathbf{X_1} = \mathbf{X_2} = \mathbf{X}$ as fuzzy numbers:

- If $\mathbf{X_1} = \mathbf{X_2}$ are same as fuzzy numbers, but are completely independent, then we should use $\mathbf{X_1} \cdot \mathbf{B} + \mathbf{X_2} \cdot \mathbf{C}$, as it ensures, that $\mathbf{X_1}$ and $\mathbf{X_2}$ can reflect different numbers in reality.
- But if $\mathbf{X_1}$ and $\mathbf{X_2}$ represent the same, we should use $\mathbf{X} \cdot (\mathbf{B} + \mathbf{C})$, so that $\mathbf{X}$ will be same for both multiplications.

3) $(\mathbf{X_1} + \mathbf{A}) \cdot \mathbf{X_2} \neq \mathbf{X^2} + \mathbf{A} \cdot \mathbf{X}$, where $\mathbf{X_1} = \mathbf{X_2} = \mathbf{X}$ as fuzzy numbers:

Unfortunately, polynomials can’t be solved so easily, since we can’t reduce usage of **X** to one place only.

But the general “thumb rule” can help reduce uncertainty:

- Factor out X as much as possible, if $\mathbf{X_1}$ and $\mathbf{X_2}$ represent the same.
- Use general (non-factorized) polynomial form, if $\mathbf{X_1}$ and $\mathbf{X_2}$ are independent.

Alternative for polynomials is solving the whole problem by computationally finding real max and min, but I don’t consider it good decision, as it contradicts one of the main advantages of fuzzy arithmetic – low complexity of computation.

On the next screen this idea will be illustrated with some examples from physics and economics.

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Example 1 (from economics):

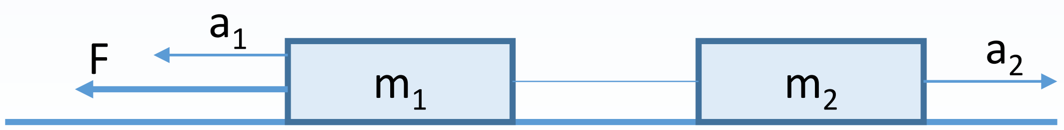
- **TR** = [80 000, 100 000] – estimated monthly revenue of the company in Euros;
- **TC** = [48 000, 60 000] = 0.6 · **TR** – estimated monthly variable costs in Euros;

1) If costs are directly related to revenue (e.g. company uses fixed margin and have no fixed costs), so that just 60% of revenue will be spent, then the best way to calculate profit is:
TP = **TR** – **TC** = **TR** – 0.6 · **TR** = 0.4 · **TR** = [32 000, 40 000] Euros (less uncertainty).

2) If the costs are completely unrelated to revenue but simple have an estimation which is numerically equal to [48 000, 60 000] (e.g. only fixed costs), then the best calculation is:
TP = **TR** – **TC** = [80 000, 100 000] – [48 000, 60 000] = [20 000, 52 000] Euros (more uncertainty).

Example 2 (from economics):

Let's consider to objects, pulling each other with a rope with no mass:



- $\mathbf{m_1}$ = [95, 105] – mass of the first object in kilograms;
- $\mathbf{m_2}$ = [85, 95] – mass of the second object in kilograms;
- $\mathbf{a_1} = \mathbf{a_2} = \mathbf{a}$ = [9, 11] – acceleration, applied to each object respectively in m/s^2.

To calculate the total force of the system, we use the formula: $\mathbf{F} = \mathbf{a_1} \cdot \mathbf{m_1} - \mathbf{a_2} \cdot \mathbf{m_2}$

1) If it is known, that applied force to both objects is exactly same, then acceleration should be factored out:
$\mathbf{F} = \mathbf{a} \cdot (\mathbf{m_1} - \mathbf{m_2})$ = [9, 11] · ([95, 105] – [85, 95]) = [9, 11] · [0, 20] = [0, 220] newtons (less uncertainty).

2) But if the acceleration is different for each object, just with the same estimated number, then it can't be factored out:
$\mathbf{F} = \mathbf{a_1} \cdot \mathbf{m_1} - \mathbf{a_2} \cdot \mathbf{m_2}$ = [9, 11] · [95, 105] – [9, 11] · [85, 95] = [855, 1155] – [765, 1045] = [-190, 390] newtons (more uncertainty).

So the results might have significant difference (up to different signs), based on the way of calculation. But such simple meaningful analysis can help draw the difference between them.

Problems with equations

The other well known problem related to the fact that subtraction is not an inverse addition (same for multiplication) is related to the fact that:

A + **X** = **B** $\not\Leftrightarrow$ **X** = **B** – **A**

A simple example would be:
If we use **X** = **B** – **A**, we get **X** = [5, 9] – [2, 4] = [1, 7].
It is easy to see, that for [2, 4] + **X** = [5, 9] the right answer is [3, 5]. The real answer is less fuzzy and uncertain, then the one above answer.

To get the right answer we can use inverse calculations, or the Kaucher intervals, where:
A + **X** = **B** $\Leftrightarrow$ **X** = **B** – **dual(A)**, where **dual(A)** = **dual(**[a1, a2]**)** = [a2, a1],
So in this case it will **X** = [5, 9] – [4, 2] = [3, 5].
While this doesn't fit classic interval arithmetic, it's rather useful for inverse computations.

But, again, we need to distinguish the difference between **A** + **X** = **B** and **X** = **B** – **A**. It is not equal, and it shouldn't be such. And the "common sense" difference is:
- If we just calculate the expression (e.g. X = B – A), we should use it straightforward without Kaucher intervals;
- But if we have an equation (e.g. A + B = X), the usage of Kaucher intervals can help to reduce uncertainty.

Conclusion

This paper proposes some simple rules to deal with problems of absence of inverse elements and distributivity, based on whether variables represent the same thing.
Such approach can reduce uncertainty in many cases, when working with fuzzy arithmetic.

In some cases, we cannot directly judge whether fuzzy numbers represent the same measurement of the same event or object. In this case it is more safe to consider them independent, as it will lead to less accurate and more fuzzy results, leading to understanding that we are less sure about the answer.

On the other hand, when dealing with collected data, we can use some help from the statistics. For example, we can calculate correlation and use some threshold to decide whether we consider these variables as dependent or not.

Speaking about further research of this topic, there is a room for development of concept of dependence in terms of fuzzy arithmetic and its influence on calculations.