How Much For an Interval? a Set? a Twin Set? a p-Box? A Kaucher Interval? Towards an Economics-Motivated Approach to Decision Making Under Uncertainty

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1. Need for Decision Making

- In many practical situations:
  - we have several alternatives, and
  - we need to select one of these alternatives.
- *Examples:*
  - a person saving for retirement needs to find the best way to invest money;
  - a company needs to select a location for its new plant;
  - a designer must select one of several possible designs for a new airplane;
  - a medical doctor needs to select a treatment for a patient.

2. Need for Decision Making Under Uncertainty

- Decision making is easier if we know the exact consequences of each alternative selection.
- Often, however:
  - we only have an incomplete information about consequences of different alternative, and
  - we need to select an alternative under this uncertainty.

3. How Decisions Under Uncertainty Are Made Now

- Traditional decision making assumes that:
  - for each alternative a,
  - we know the probability $p_i(a)$ of different outcomes i.
- It can be proven that:
  - preferences of a rational decision maker can be described by *utilities* u_i so that
  - an alternative a is better if its expected utility $\overline{u}(a) \stackrel{\text{def}}{=} \sum_i p_i(a) \cdot u_i$ is larger.

4. Hurwicz Optimism-Pessimism Criterion

- Often, we do not know these probabilities p_i.
- For example, sometimes:
  - we only know the range $[\underline{u}, \overline{u}]$ of possible utility values, but
  - we do not know the probability of different values within this range.
- It has been shown that in this case, we should select an alternative s.t. $\alpha_H \cdot \overline{u} + (1 - \alpha_H) \cdot \underline{u} \to \max$.
- Here, $\alpha_H \in [0, 1]$ described the optimism level of a decision maker:
  - $\alpha_H = 1$ means optimism;
  - $\alpha_H = 0$ means pessimism;
  - $0 < \alpha_H < 1$ combines optimism and pessimism.

5. Fair Price Approach: An Idea

- When we have a full information about an object, then:
  - we can express our desirability of each possible situation
  - by declaring a price that we are willing to pay to get involved in this situation.
- Once these prices are set, we simply select the alternative for which the participation price is the highest.
- In decision making under uncertainty, it is not easy to come up with a fair price.
- A natural idea is to develop techniques for producing such fair prices.
- These prices can then be used in decision making, to select an appropriate alternative.

6. Case of Interval Uncertainty

- *Ideal case:* we know the exact gain u of selecting an alternative.
- *A more realistic case:* we only know the lower bound $\underline{u}$ and the upper bound $\overline{u}$ on this gain.
- *Comment:* we do not know which values $u \in [\underline{u}, \overline{u}]$ are more probable or less probable.
- This situation is known as *interval uncertainty.*
- We want to assign, to each interval $[\underline{u}, \overline{u}]$, a number $P([\underline{u}, \overline{u}])$ describing the fair price of this interval.
- Since we know that $u \le \overline{u}$, we have $P([\underline{u}, \overline{u}]) \le \overline{u}$.
- Since we know that $\underline{u}$, we have $\underline{u} \le P([\underline{u}, \overline{u}])$.

7. Case of Interval Uncertainty: Monotonicity

- *Case 1:* we keep the lower endpoint $\underline{u}$ intact but increase the upper bound.
- This means that we:
  - keeping all the previous possibilities, but
  - we allow new possibilities, with a higher gain.
- In this case, it is reasonable to require that the corresponding price not decrease:

$$\text{if } \underline{u} = \underline{v} \text{ and } \overline{u} < \overline{v} \text{ then } P([\underline{u}, \overline{u}]) \le P([\underline{v}, \overline{v}]).$$

- *Case 2:* we dismiss some low-gain alternatives.
- This should increase (or at least not decrease) the fair price:

$$\text{if } \underline{u} < \underline{v} \text{ and } \overline{u} = \overline{v} \text{ then } P([\underline{u}, \overline{u}]) \le P([\underline{v}, \overline{v}]).$$

8. Additivity: Idea

- Let us consider the situation when we have two consequent independent decisions.
- We can consider two decision processes separately.
- We can also consider a single decision process in which we select a pair of alternatives:
  - the 1st alternative corr. to the 1st decision, and
  - the 2nd alternative corr. to the 2nd decision.
- If we are willing to pay:
  - the amount u to participate in the first process, and
  - the amount v to participate in the second decision process,
- then we should be willing to pay $u + v$ to participate in both decision processes.

9. Additivity: Case of Interval Uncertainty

- About the gain u from the first alternative, we only know that this (unknown) gain is in $[\underline{u}, \overline{u}]$.
- About the gain v from the second alternative, we only know that this gain belongs to the interval $[\underline{v}, \overline{v}]$.
- The overall gain $u + v$ can thus take any value from the interval

$$[\underline{u}, \overline{u}] + [\underline{v}, \overline{v}] \stackrel{\text{def}}{=} \{u + v : u \in [\underline{u}, \overline{u}], v \in [\underline{v}, \overline{v}]\}.$$

- It is easy to check that

$$[\underline{u}, \overline{u}] + [\underline{v}, \overline{v}] = [\underline{u} + \underline{v}, \overline{u} + \overline{v}].$$

- Thus, the additivity requirement about the fair prices takes the form

$$P([\underline{u} + \underline{v}, \overline{u} + \overline{v}]) = P([\underline{u}, \overline{u}]) + P([\underline{v}, \overline{v}]).$$

10. Fair Price Under Interval Uncertainty

- By a *fair price under interval uncertainty*, we mean a function $P([\underline{u}, \overline{u}])$ for which:
  - $\underline{u} \le P([\underline{u}, \overline{u}]) \le \overline{u}$ for all $\underline{u}$ and $\overline{u}$ (*conservativeness*);
  - if $\underline{u} = \underline{v}$ and $\overline{u} < \overline{v}$, then $P([\underline{u}, \overline{u}]) \le P([\underline{v}, \overline{v}])$ (*monotonicity*);
  - (*additivity*) for all $\underline{u}$, $\overline{u}$, $\underline{v}$, and $\overline{v}$, we have

$$P([\underline{u} + \underline{v}, \overline{u} + \overline{v}]) = P([\underline{u}, \overline{u}]) + P([\underline{v}, \overline{v}]).$$

- *Theorem:* Each fair price under interval uncertainty has the form

$$P([\underline{u}, \overline{u}]) = \alpha_H \cdot \overline{u} + (1 - \alpha_H) \cdot \underline{u} \text{ for some } \alpha_H \in [0, 1].$$

- *Comment:* we thus get a new justification of Hurwicz optimism-pessimism criterion.

11. Proof: Main Ideas

- Due to monotonicity, $P([u,u]) = u$.
- Due to monotonicity, $\alpha_H \stackrel{\text{def}}{=} P([0,1]) \in [0,1]$.
- For $[0,1] = [0,1/n] + \ldots + [0,1/n]$ (n times), additivity implies $\alpha_H = n \cdot P([0,1/n])$, so $P([0,1/n]) = \alpha_H \cdot (1/n)$.
- For $[0,m/n] = [0,1/n] + \ldots + [0,1/n]$ (m times), additivity implies $P([0,m/n]) = \alpha_H \cdot (m/n)$.
- For each real number r, for each n, there is an m s.t. $m/n \le r \le (m+1)/n$.
- Monotonicity implies $\alpha_H \cdot (m/n) = P([0,m/n]) \le P([0,r]) \le P([0,(m+1)/n]) = \alpha_H \cdot ((m+1)/n)$.
- When $n \to \infty$, $\alpha_H \cdot (m/n) \to \alpha_H \cdot r$ and $\alpha_H \cdot ((m+1)/n) \to \alpha_H \cdot r$, hence $P([0,r]) = \alpha_H \cdot r$.
- For $[\underline{u},\overline{u}] = [\underline{u},\underline{u}] + [0,\overline{u}-\underline{u}]$, additivity implies $P([\underline{u},\overline{u}]) = \underline{u} + \alpha_H \cdot (\overline{u}-\underline{u})$. Q.E.D.

12. Case of Set-Valued Uncertainty

- In some cases:
  - in addition to knowing that the actual gain belongs to the interval $[\underline{u}, \overline{u}]$,
  - we also know that some values from this interval cannot be possible values of this gain.
- For example:
  - if we buy an obscure lottery ticket for a simple prize-or-no-prize lottery from a remote country,
  - we either get the prize or lose the money.
- In this case, the set of possible values of the gain consists of two values.
- Instead of a (bounded) *interval* of possible values, we can consider a general bounded *set* of possible values.

13. Fair Price Under Set-Valued Uncertainty

- We want a function P that assigns, to every bounded closed set S, a real number $P(S)$, for which:
  - $P([\underline{u},\overline{u}]) = \alpha_H\cdot\overline{u}+(1-\alpha_H)\cdot\underline{u}$ (*conservativeness*);
  - $P(S+S') = P(S)+P(S')$, where $S+S' \stackrel{\text{def}}{=} \{s+s' : s\in S, s'\in S'\}$ (*additivity*).
- *Theorem:* Each fair price under set uncertainty has the form $P(S)=\alpha_H\cdot\sup S+(1-\alpha_H)\cdot\inf S$.
- *Proof: idea.*
  - $\{\underline{s},\overline{s}\}\subseteq S\subseteq[\underline{s},\overline{s}]$, where $\underline{s}\stackrel{\text{def}}{=}\inf S$ and $\underline{s}\stackrel{\text{def}}{=}\sup S$;
  - thus, $[2\underline{s},2\overline{s}]=\{\underline{s},\overline{s}\}+[\underline{s},\overline{s}]\subseteq S+[\underline{s},\overline{s}]\subseteq [\underline{s},\overline{s}]+[\underline{s},\overline{s}]=[2\underline{s},2\overline{s}]$;
  - so $S+[\underline{s},\overline{s}]=[2\underline{s},2\overline{s}]$, hence $P(S)+P([\underline{s},\overline{s}])=P([2\underline{s},2\overline{s}])$, and

$$P(S)=(\alpha_H\cdot(2\overline{s})+(1-\alpha_H)\cdot(2\underline{s}))-(\alpha_H\cdot\overline{s}+(1-\alpha_H)\cdot\underline{s}).$$

14. Case of Probabilistic Uncertainty

- Suppose that for some financial instrument, we know a prob. distribution $\rho(x)$ on the set of possible gains x.
- What is the fair price P for this instrument?
- Due to additivity, the fair price for n copies of this instrument is $n \cdot P$.
- According to the Large Numbers Theorem, for large n, the average gain tends to the mean value

$$\mu = \int x \cdot \rho(x)\, dx.$$

- Thus, the fair price for n copies of the instrument is close to $n \cdot \mu$: $n \cdot P \approx n \cdot \mu$.
- The larger n, the closer the averages. So, in the limit, we get $P = \mu$.

15. Case of p-Box Uncertainty

- Probabilistic uncertainty means that for every x, we know the value of the cdf $F(x) = \text{Prob}(\eta \le x)$.
- In practice, we often only have partial information about these values.
- In this case, for each x, we only know an interval $[\underline{F}(x), \overline{F}(x)]$ containing the actual (unknown) value $F(x)$.
- The interval-valued function $[\underline{F}(x), \overline{F}(x)]$ is known as a *p-box*.
- What is the fair price of a p-box?
- The only information that we have about the cdf is that $F(x) \in [\underline{F}(x), \overline{F}(x)]$.
- For each possible $F(x)$, for large n, n copies of the instrument are $\approx$ equivalent to $n \cdot \mu$, w/ $\mu = \int x\, dF(x)$.

16. Case of p-Box Uncertainty (cont-d)

- For each possible $F(x)$, for large n, n copies of the instrument are $\approx$ equivalent to $n \cdot \mu$, where

$$\mu = \int x\, dF(x).$$

- For different $F(x)$, values of μ for an interval $[\underline{\mu}, \overline{\mu}]$, where $\underline{\mu} = \int x\, d\overline{F}(x)$ and $\overline{\mu} = \int x\, d\underline{F}(x)$.
- Thus, the price of a p-box is equal to the price of an interval $[\underline{\mu}, \overline{\mu}]$.
- We already know that this price is equal to

$$\alpha_H \cdot \overline{\mu} + (1 - \alpha_H) \cdot \underline{\mu}.$$

- So, this is a fair price of a p-box.

17. Case of Kaucher (Improper) Intervals

- What is the price for an improper interval $[\underline{x}, \overline{x}]$, with $\underline{x} > \overline{x}$?
- Let us use additivity; here:

$$[\underline{x}, \overline{x}] + [\overline{x}, \underline{x}] = [\underline{x} + \overline{x}, \underline{x} + \overline{x}].$$

- Thus,

$$P([\underline{x}, \overline{x}]) + P([\overline{x}, \underline{x}]) = P([\underline{x} + \overline{x}, \underline{x} + \overline{x}]).$$

- We know that $P([\overline{x}, \underline{x}]) = \alpha_H \cdot \underline{x} + (1 - \alpha_H) \cdot \overline{x}$ and $P(\underline{x} + \overline{x}) = \underline{x} + \overline{x}$; hence:

$$P([\underline{x}, \overline{x}]) = (\underline{x} + \overline{x}) - (\alpha_H \cdot \underline{x} + (1 - \alpha_H) \cdot \overline{x}).$$

- Therefore, $P([\underline{x}, \overline{x}]) = \alpha_H \cdot \overline{x} + (1 - \alpha_H) \cdot \underline{x}$.

18. Case of Triples

- Sometimes, in addition to an interval $[\underline{x}, \overline{x}]$, we also have a "most probable" value x within this interval.

- For such triples, addition is defined component-wise:

$$([\underline{x}, \overline{x}], x) + ([\underline{y}, \overline{y}], y) = ([\underline{x} + \underline{y}, \overline{x} + \overline{y}], x + y).$$

- Thus, the additivity for additivity requirement about the fair prices takes the form

$$P([\underline{x} + \underline{y}, \overline{x} + \overline{y}], x + y) = P([\underline{x}, \overline{x}], x) + P([\underline{y}, \overline{y}], y).$$

19. Fair Price Under Triple Uncertainty

- By a *fair price under triple uncertainty*, we mean a function $P([\underline{u},\overline{u}],u)$ for which:
  - $\underline{u}\le P([\underline{u},\overline{u}],u)\le \overline{u}$ for all $\underline{u}\le u\le \overline{u}$ (*conservativeness*);
  - if $\underline{u}\le \underline{v}$, $u\le v$, and $\overline{u}\le \overline{v}$, then $P([\underline{u},\overline{u}],u)\le P([\underline{v},\overline{v}],v)$ (*monotonicity*);
  - (*additivity*) for all $\underline{u}$, $\overline{u}$, u $\underline{v}$, $\overline{v}$, and v, we have

$$P([\underline{u}+\underline{v},\overline{u}+\overline{v}],u+v)=P([\underline{u},\overline{u}],u)+P([\underline{v},\overline{v}],v).$$

- *Theorem:* Each fair price under triple uncertainty has the form

$$P([\underline{u},\overline{u}],u)=\alpha_L\cdot\underline{u}+(1-\alpha_L-\alpha_U)\cdot u+\alpha_U\cdot\overline{u},\ \text{ where } \alpha_L,\alpha_U\in[0,1].$$

20. Fair Price Under Triple Uncertainty: Proof

- In general, we have

$$([\underline{u}, \overline{u}], u) = ([u, u], u) + ([0, \underline{u} - u], 0) + ([\underline{u} - u, 0], 0).$$

- So, due to additivity:

$$P([\underline{u}, \overline{u}], u) = P([u, u], u) + P([0, \underline{u} - u], 0) + P([\underline{u} - u, 0], 0).$$

- Due to conservativeness, $P([u, u], u) = u$.
- Similarly to the interval case, we can prove that $P([0, r], 0) = \alpha_U \cdot r$ for some $\alpha_U \in [0, 1]$.
- Similarly, $P([r, 0], 0) = \alpha_L \cdot r$ for some $\alpha_L \in [0, 1]$.
- Thus,

$$P([\underline{u}, \overline{u}], u) = \alpha_L \cdot \underline{u} + (1 - \alpha_L - \alpha_U) \cdot u + \alpha_U \cdot \overline{u}.$$

21. Case of Twin Intervals

- Sometimes, instead of a "most probable" value x, we have a "most probable" subinterval $[\underline{m}, \overline{m}] \subseteq [\underline{x}, \overline{x}]$.
- For such "twin intervals", addition is defined component-wise:

$$([\underline{x}, \overline{x}], [\underline{m}, \overline{m}])+([\underline{y}, \overline{y}], [\underline{n}, \overline{n}]) = ([\underline{x}+\underline{y}, \overline{x}+\overline{y}], [\underline{m}+\underline{n}, \overline{m}+\overline{n}]).$$

- Thus, the additivity for additivity requirement about the fair prices takes the form

$$P([\underline{x}+\underline{y}, \overline{x}+\overline{y}], [\underline{m}+\underline{n}, \overline{m}+\overline{n}]) =$$
$$P([\underline{x}, \overline{x}], [\underline{m}, \overline{m}]) + P([\underline{y}, \overline{y}], [\underline{n}, \overline{n}]).$$

22. Fair Price Under Twin Interval Uncertainty

- By a *fair price under twin uncertainty*, we mean a function $P([\underline{u},\overline{u}],[\underline{m},\overline{m}])$ for which:
  - $\underline{u} \le P([\underline{u},\overline{u}],[\underline{m},\overline{m}]) \le \overline{u}$ for all $\underline{u} \le \underline{m} \le \overline{m} \le \overline{u}$ (*conservativeness*);
  - if $\underline{u} \le \underline{v}$, $\underline{m} \le \underline{n}$, $\overline{m} \le \overline{n}$, and $\overline{u} \le \overline{v}$, then $P([\underline{u},\overline{u}],[\underline{m},\overline{m}]) \le P([\underline{v},\overline{v}],[\underline{n},\overline{n}])$ (*monotonicity*);
  - for all $\underline{u} \le \underline{m} \le \overline{m} \le \overline{u}$ and $\underline{v} \le \underline{n} \le \overline{n} \le \overline{v}$, we have *additivity*:

$$P([\underline{u}+\underline{v},\overline{u}+\overline{v}],[\underline{m}+\underline{n},\overline{m}+\overline{m}]) = P([\underline{u},\overline{u}],[\underline{m},\overline{m}])+P([\underline{v},\overline{v}],[\underline{n},\overline{n}]).$$

- *Theorem:* Each fair price under twin uncertainty has the following form, for some $\alpha_L, \alpha_u, \alpha_U \in [0,1]$:

$$P([\underline{u},\overline{u}],[\underline{m},\overline{m}]) = \underline{m}+\alpha_u\cdot(\overline{m}-\underline{m})+\alpha_U\cdot(\overline{U}-\overline{m})+\alpha_L\cdot(\underline{u}-\underline{m}).$$

23. Fair Price Under Twin Uncertainty: Proof

- In general, we have

$$([\underline{u},\overline{u}],[\underline{m},\overline{m}]) = ([\underline{m},\underline{m}],[\underline{m},\underline{m}]) + ([0,\overline{m}-\underline{m}],[0,\overline{m}-\underline{m}]) + ([0,\overline{u}-\overline{m}],[0,0]) + ([\underline{u}-\underline{m},0],[0,0)].$$

- So, due to additivity:

$$P([\underline{u},\overline{u}],[\underline{m},\overline{m}]) = P([\underline{m},\underline{m}],[\underline{m},\underline{m}]) + P([0,\overline{m}-\underline{m}],[0,\overline{m}-\underline{m}]) + P([0,\overline{u}-\overline{m}],[0,0]) + P([\underline{u}-\underline{m},0],[0,0)].$$

- Due to conservativeness, $P([\underline{m},\underline{m}],[\underline{m},\underline{m}]) = \underline{m}$.
- Similarly to the interval case, we can prove that:
    - $P([0,r],[0,r]) = \alpha_u \cdot r$ for some $\alpha_u \in [0,1]$,
    - $P([0,r],[0,0]) = \alpha_U \cdot r$ for some $\alpha_U \in [0,1]$;
    - $P([r,0],[0,0]) = \alpha_L \cdot r$ for some $\alpha_L \in [0,1]$.
- Thus,

$$P([\underline{u},\overline{u}],[\underline{m},\overline{m}]) = \underline{m} + \alpha_u \cdot (\overline{m}-\underline{m}) + \alpha_U \cdot (\overline{U}-\overline{m}) + \alpha_L \cdot (\underline{u}-\underline{m}).$$

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