On the sets of H- and D-continuous interval functions

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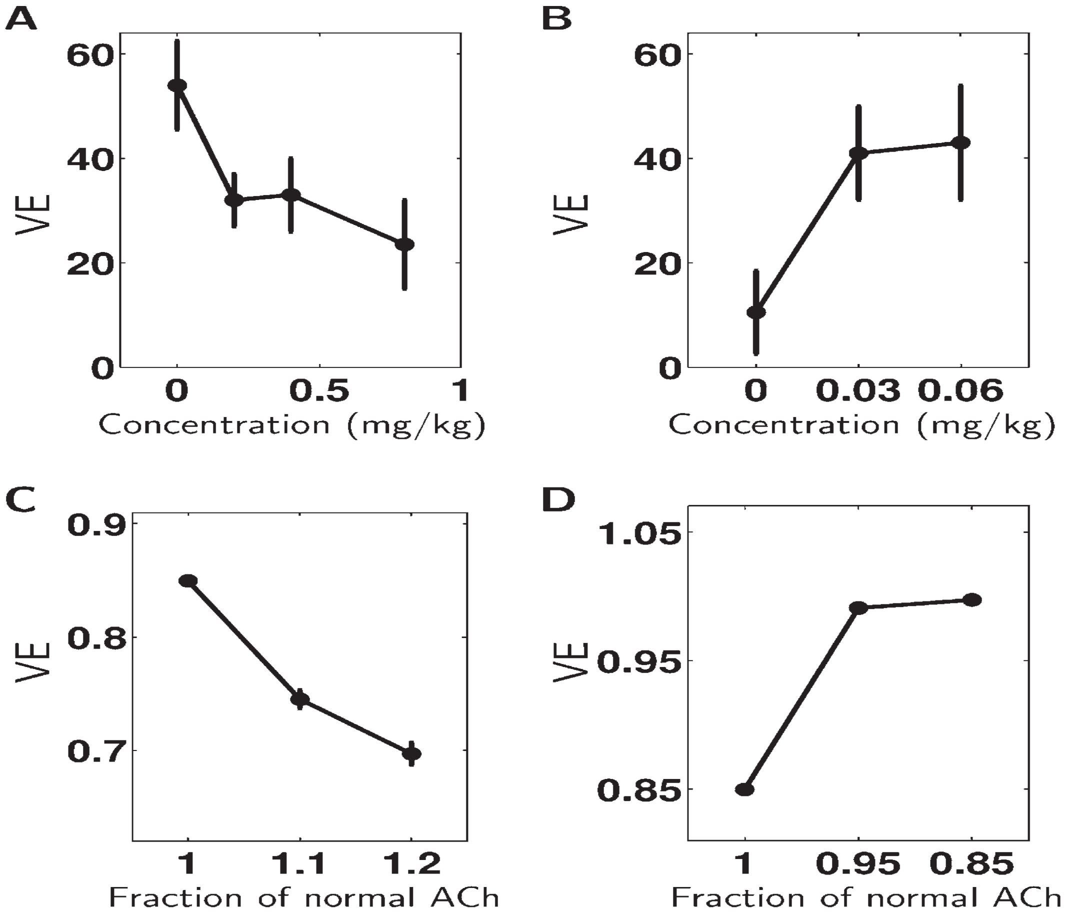
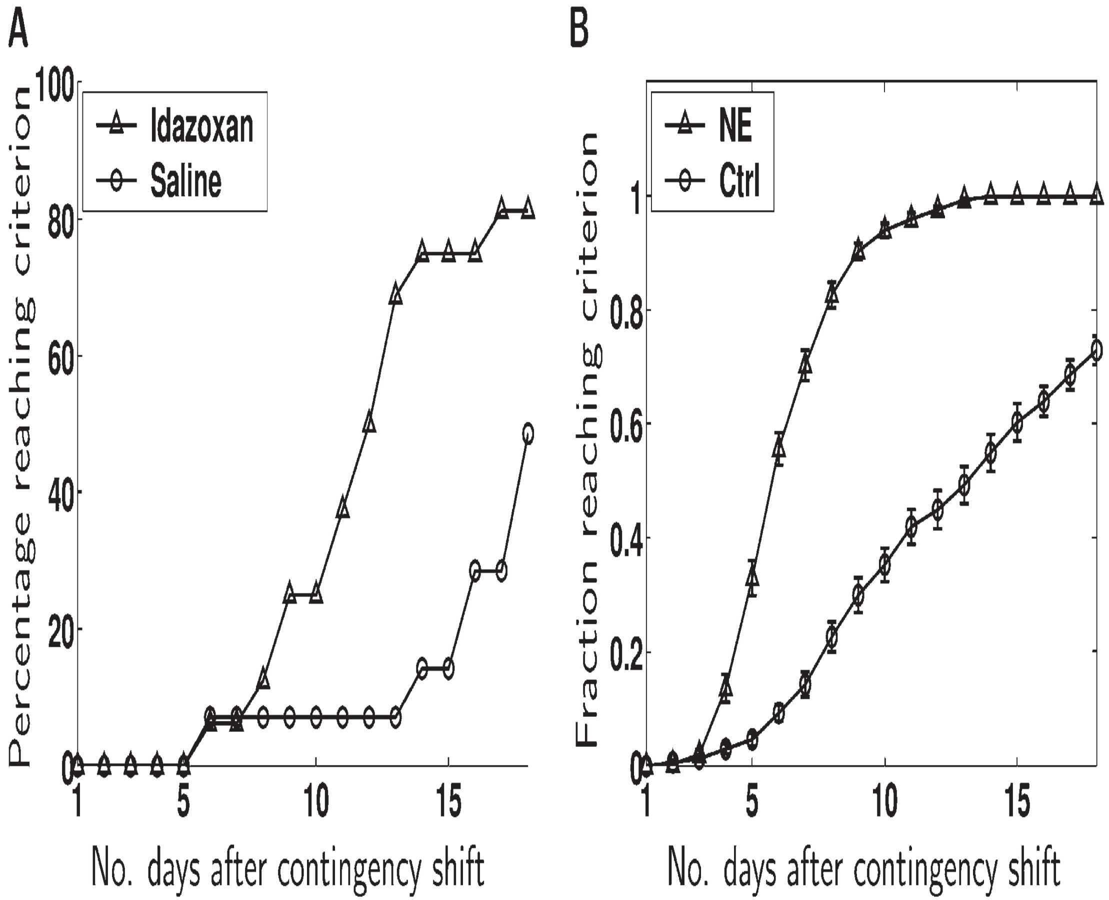
Biological problems: sensitive and uncertain input data



Biological problems: sensitive and uncertain input data



Biological problems: sensitive behaviour (jumps)

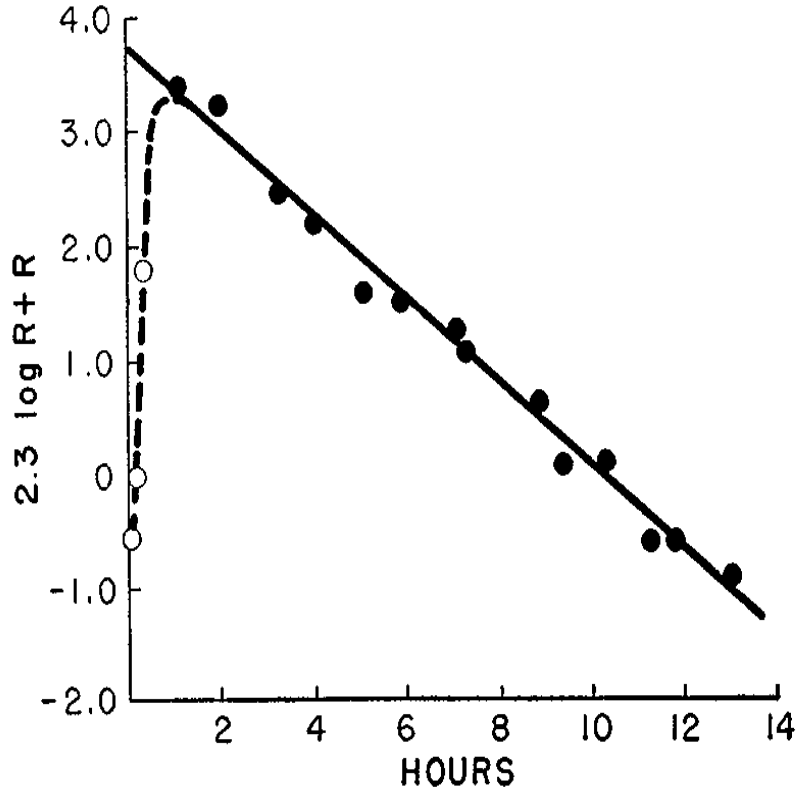


FIG. 1. The data of Experiment I are plotted in accordance with Equation 7. $R = (E')/(E)$

Biological problems: sensitive behaviour (jumps)

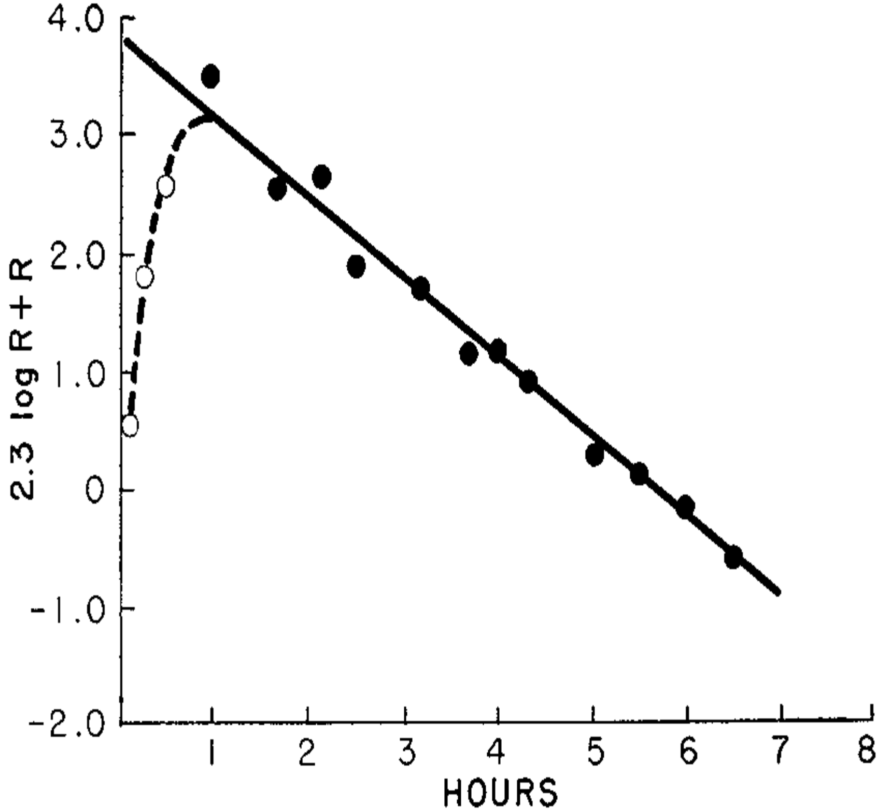


FIG. 2. The data of Experiment II are plotted in accordance with Equation 7. $R = (E')/(E)$

Biological problems: sensitive behaviour (jumps)

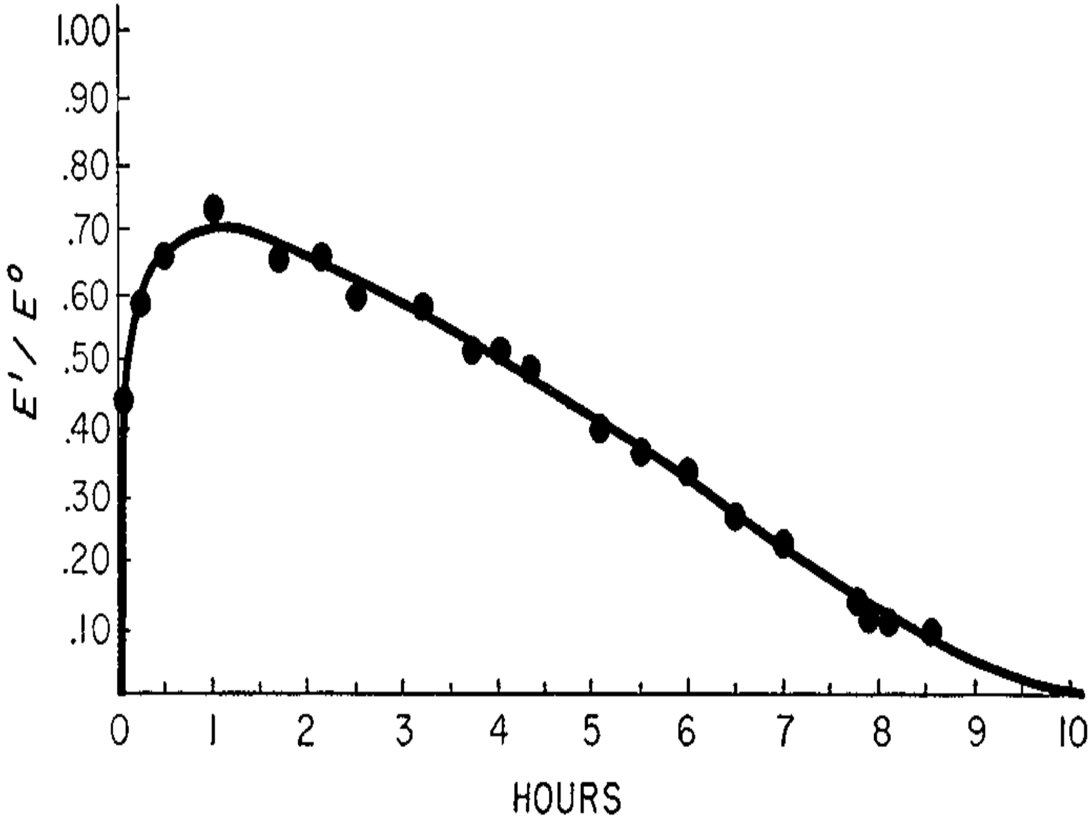
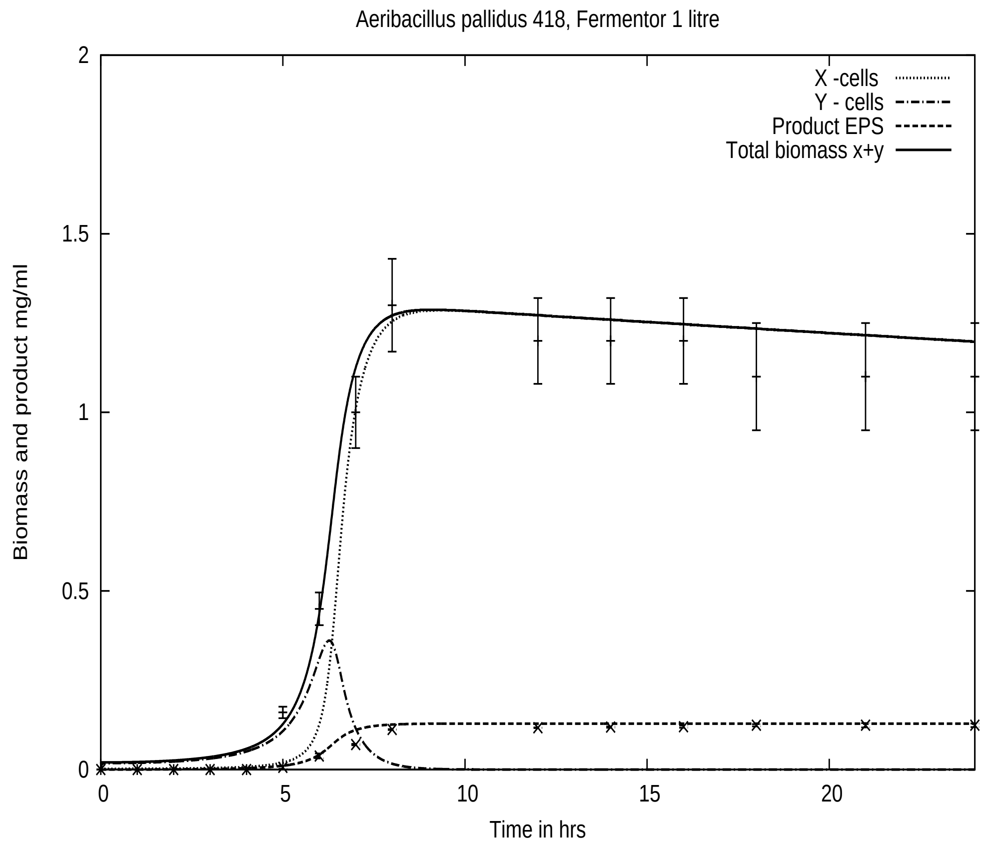
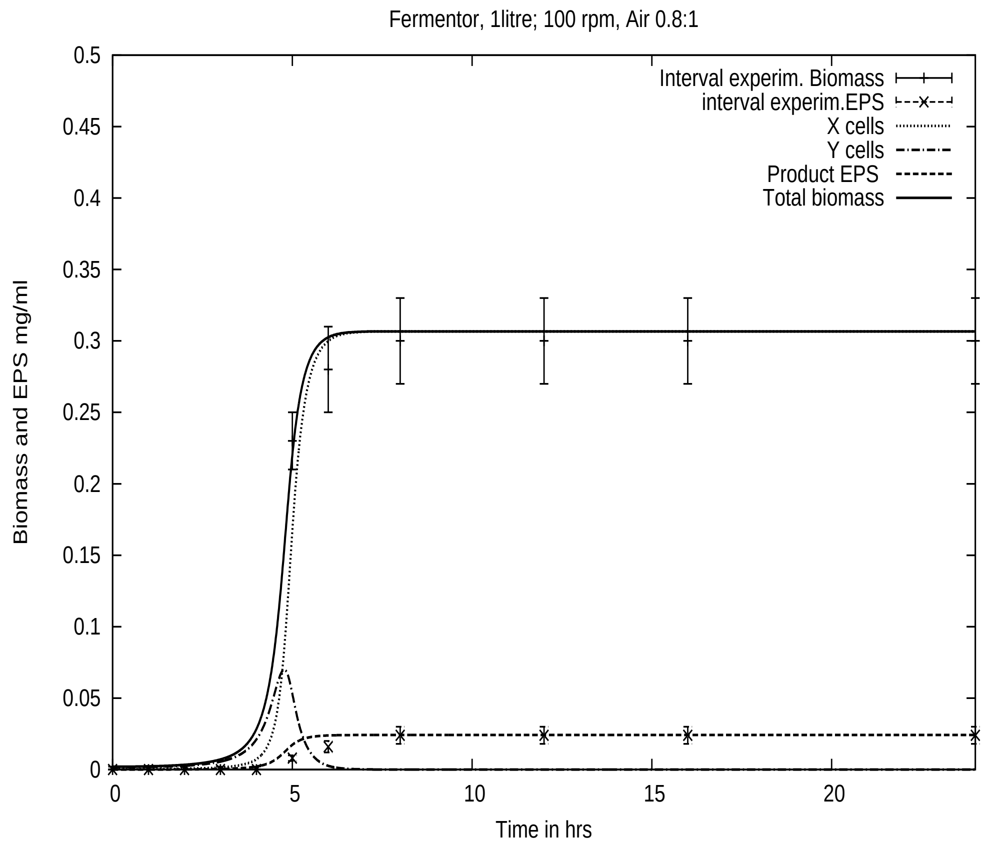


FIG. 3. The data of Experiment II are plotted in accordance with Equation 6.

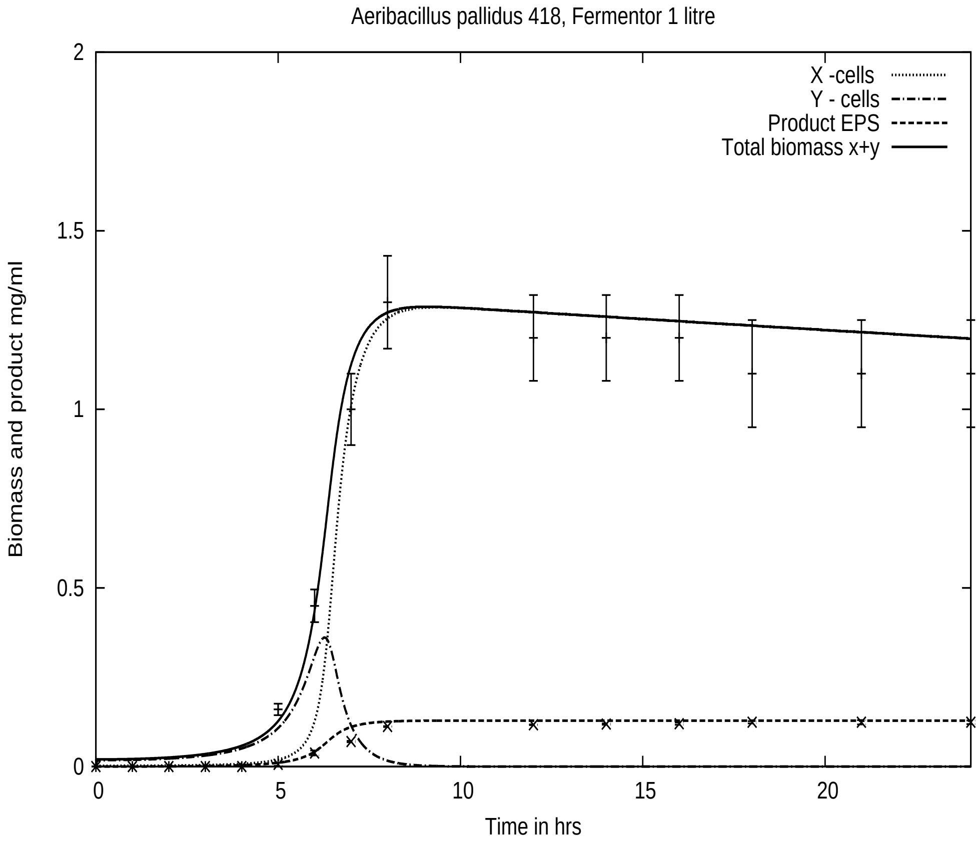
Biological problems: sensitive and uncertain input data



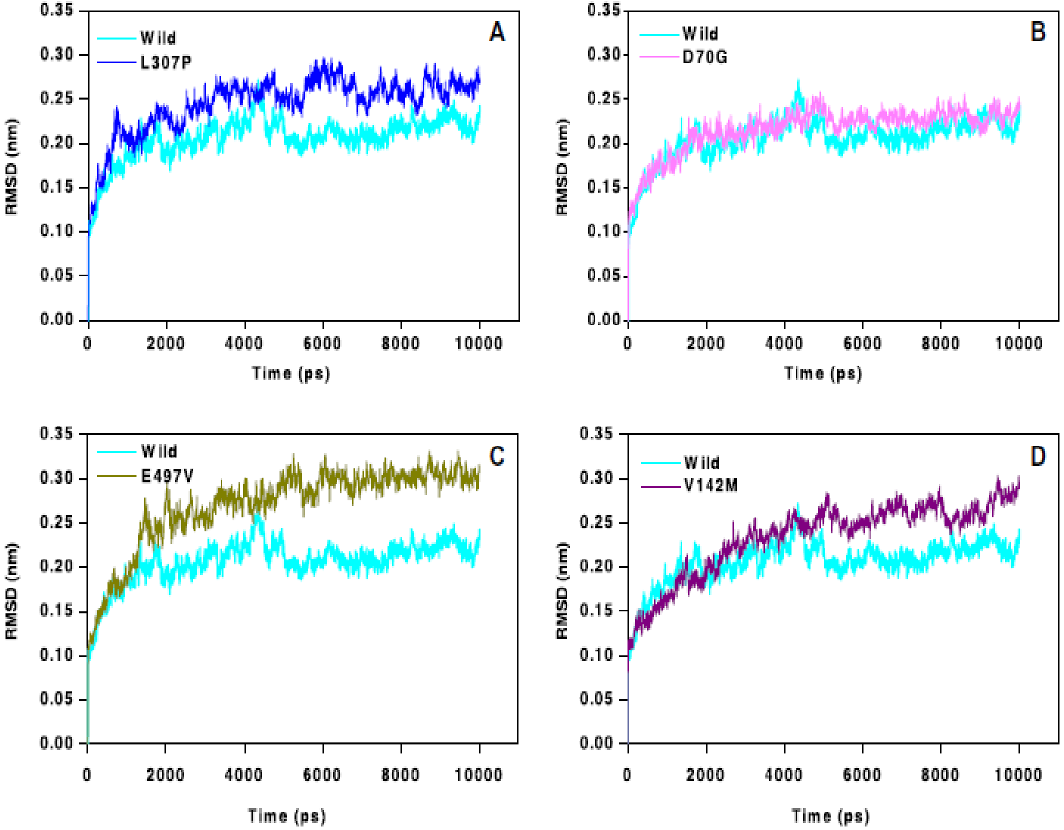
Biological problems: sensitivity and uncertainty



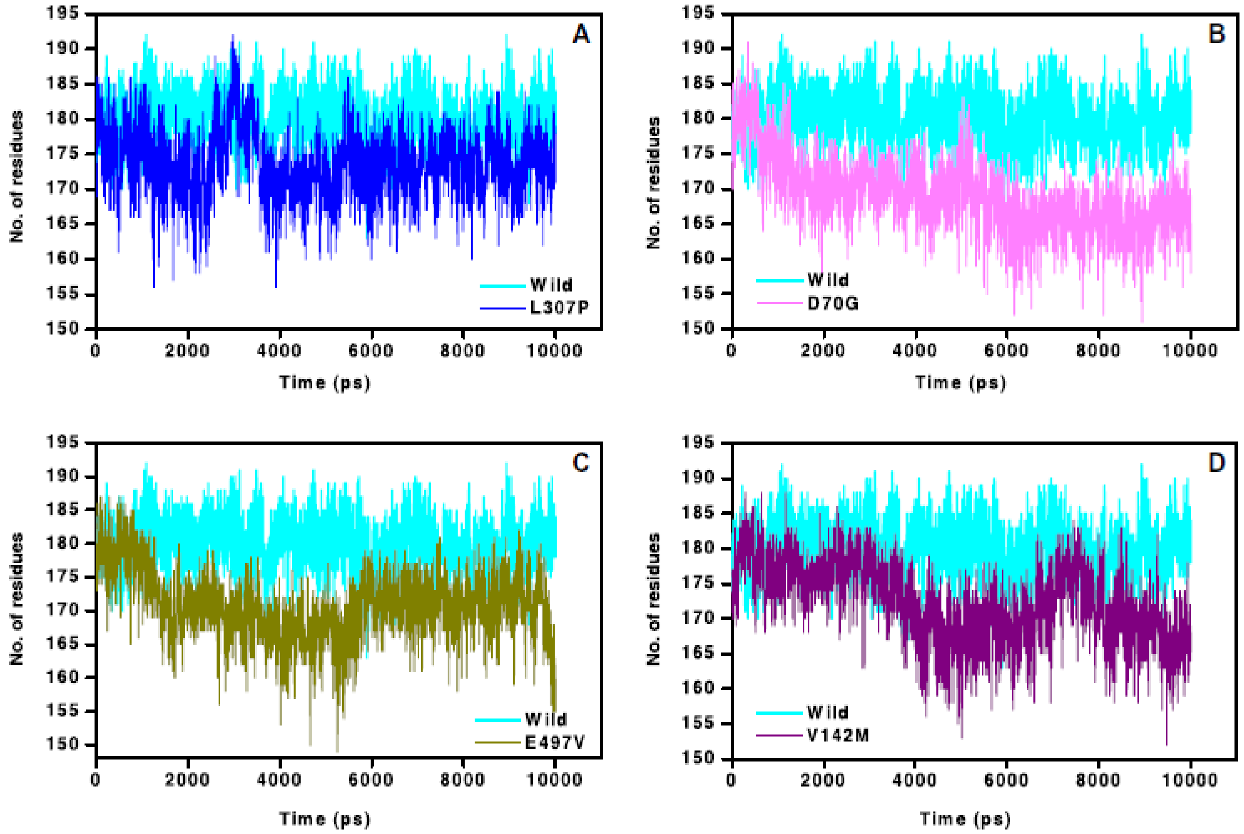
Biological problems: sensitive and uncertain input data



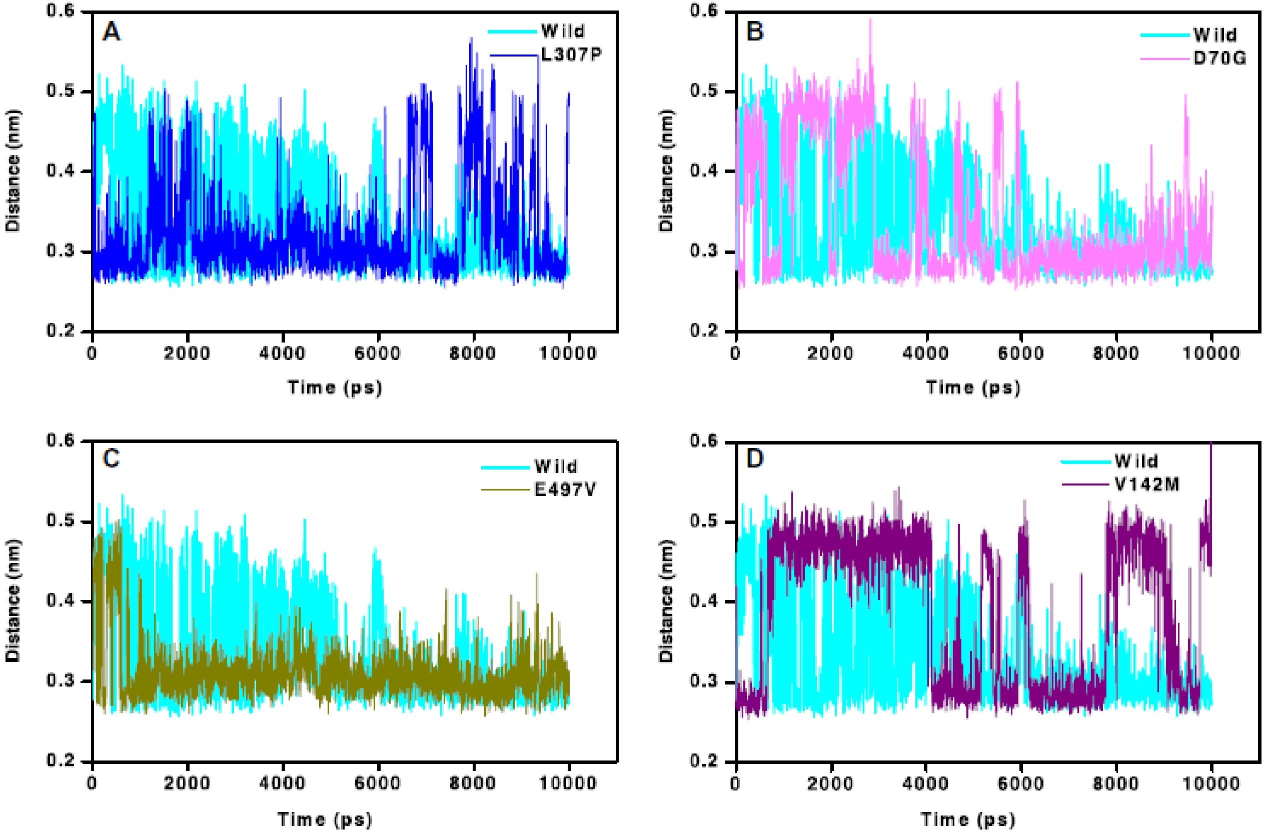
Biological problems: sensitivity and interval output data



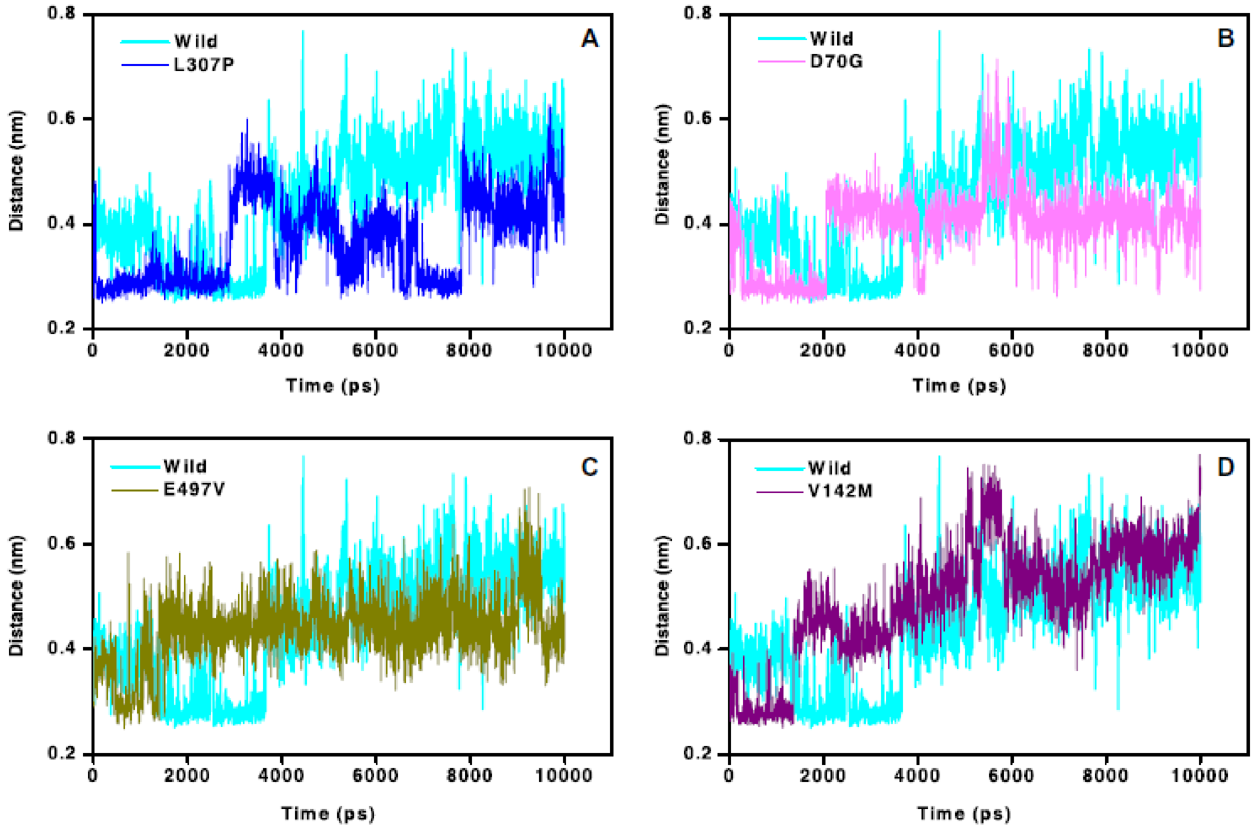
Biological problems: sensitivity and interval output data



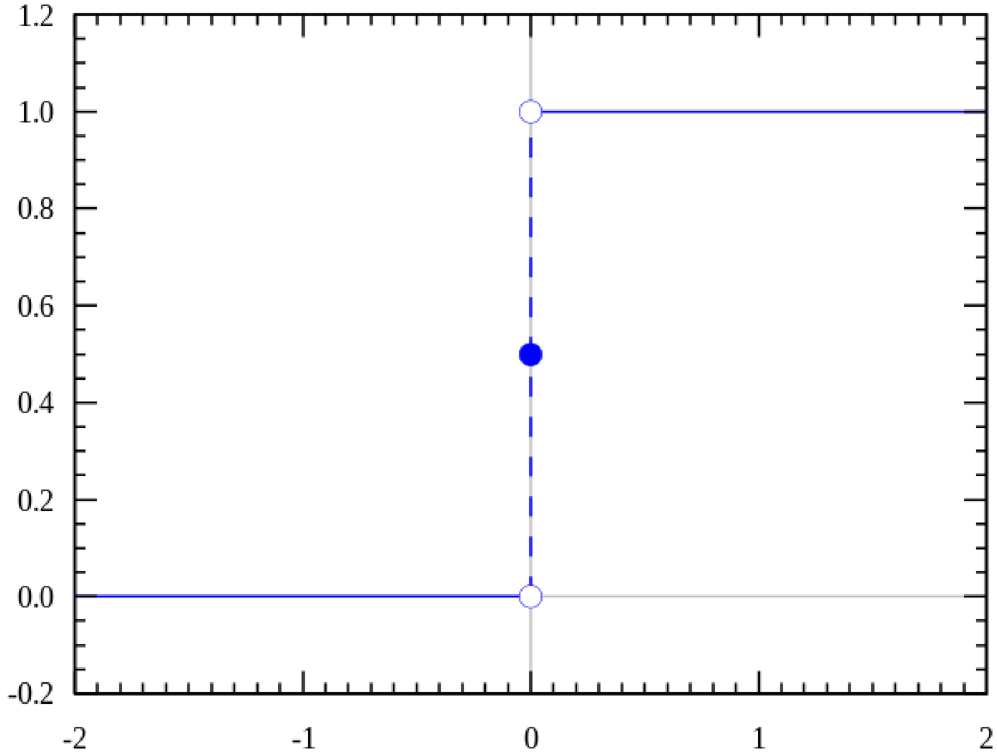
Biological problems: sensitivity and interval output data



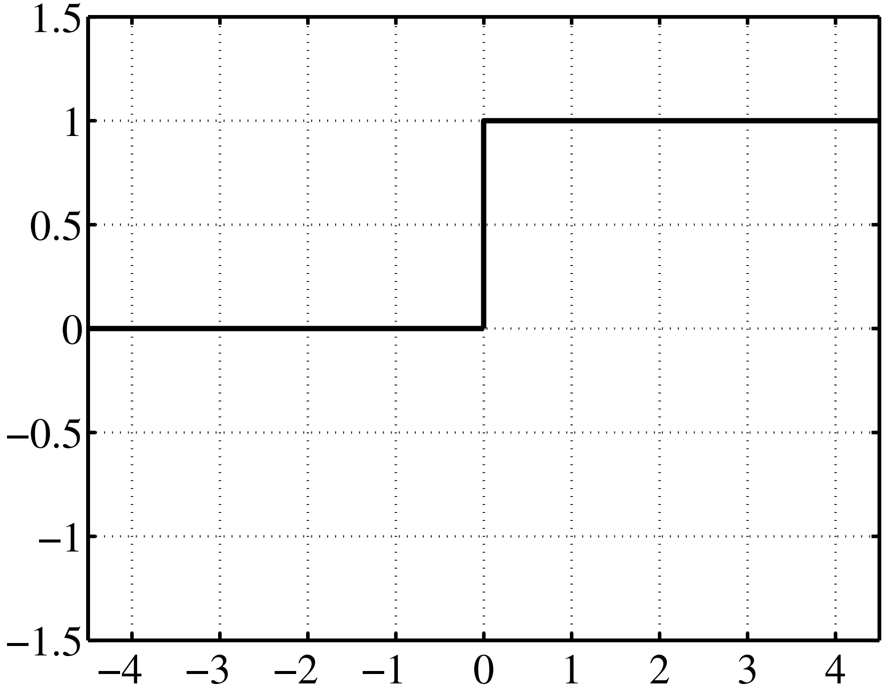
Biological problems: sensitivity and interval output data



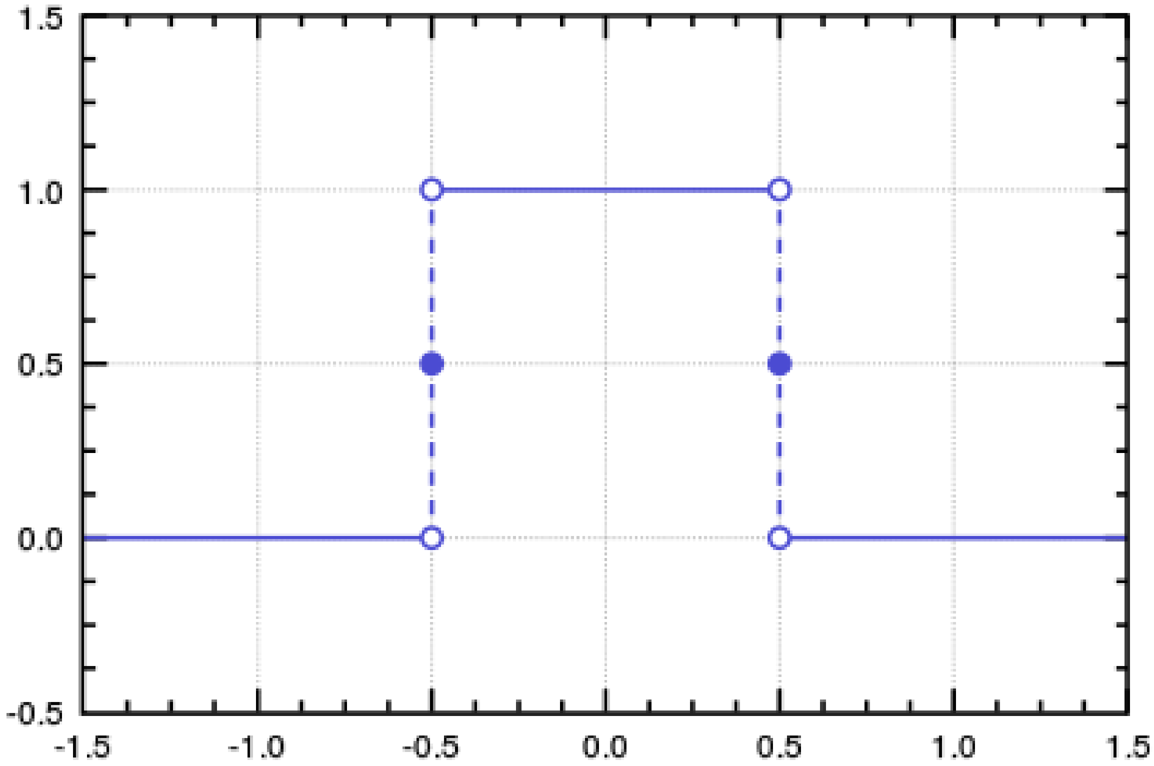
Heaviside step function as real (single-valued) function (wikipedia)



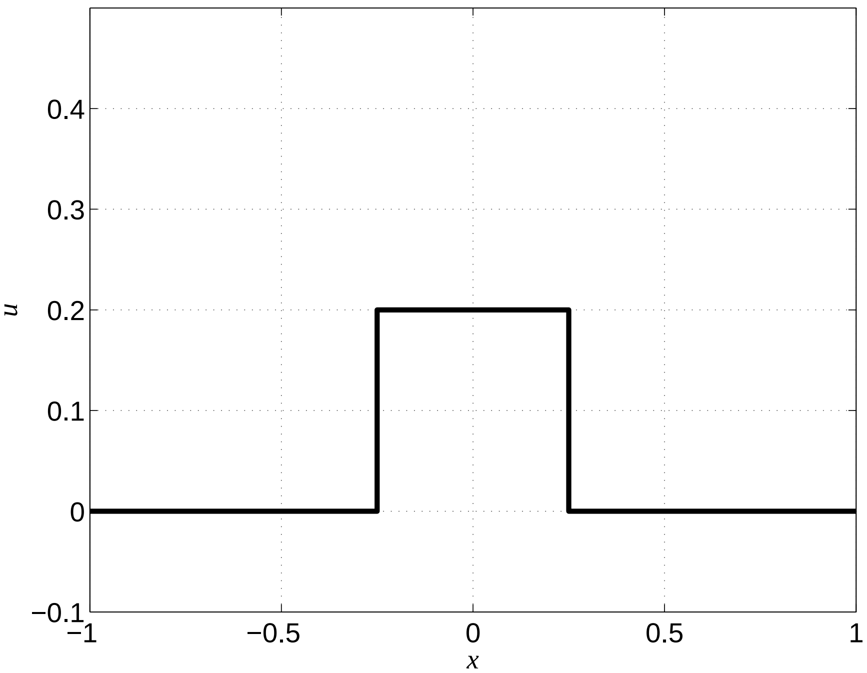
H-continuous interval step function—complete graph



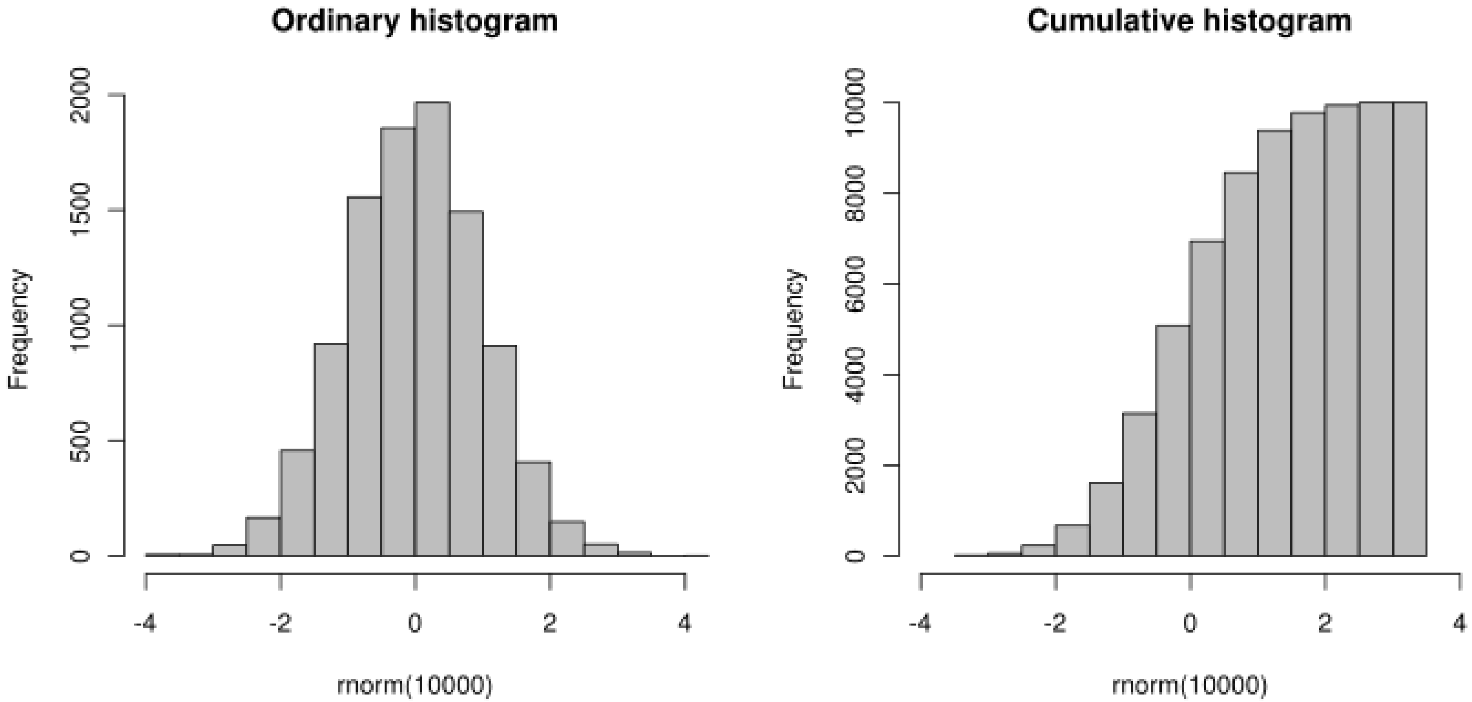
Step function as real function (wikipedia)



Step function as interval function



His


Ordinary histogram
Frequency
0
500
1000
1500
2000
-4
-2
0
2
4
rnorm(10000)
Cumulative histogram
Frequency
0
2000
4000
6000
8000
10000
-4
-2
0
2
4
rnorm(10000)

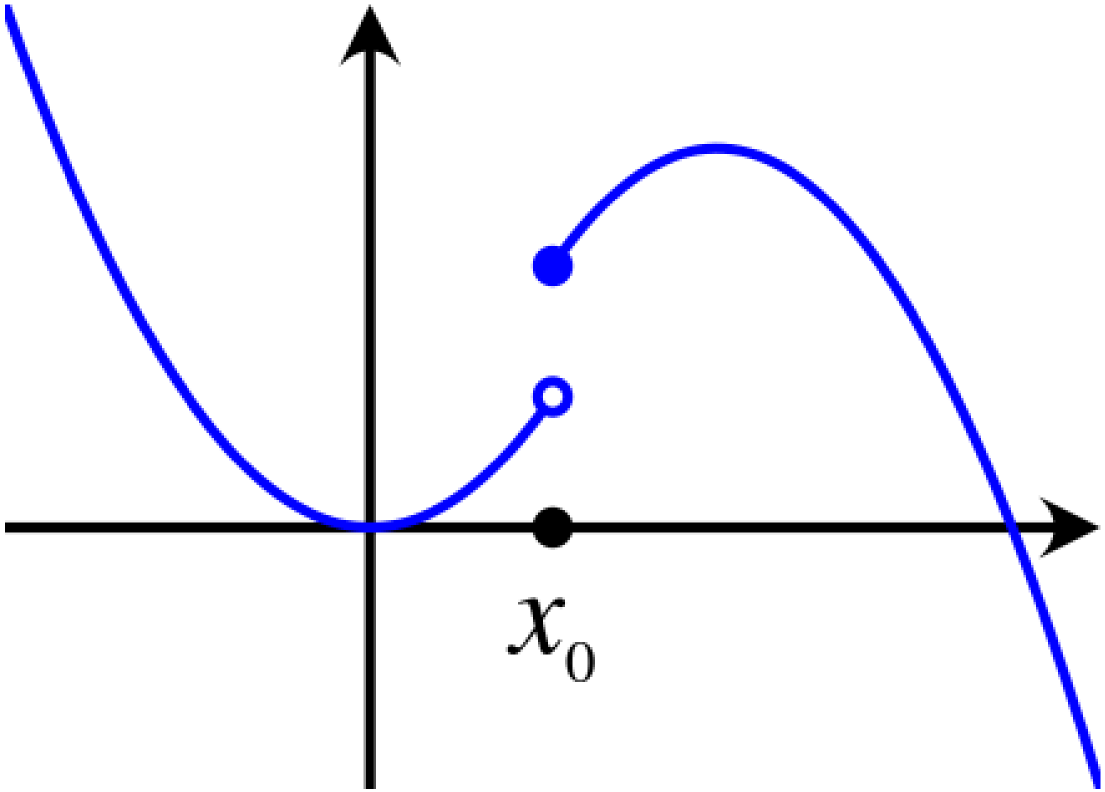
Hausdorff continuity

The concept of Hausdorff continuity generalizes the concept of continuity in such a way that many essential properties of the usual continuous real functions are preserved.

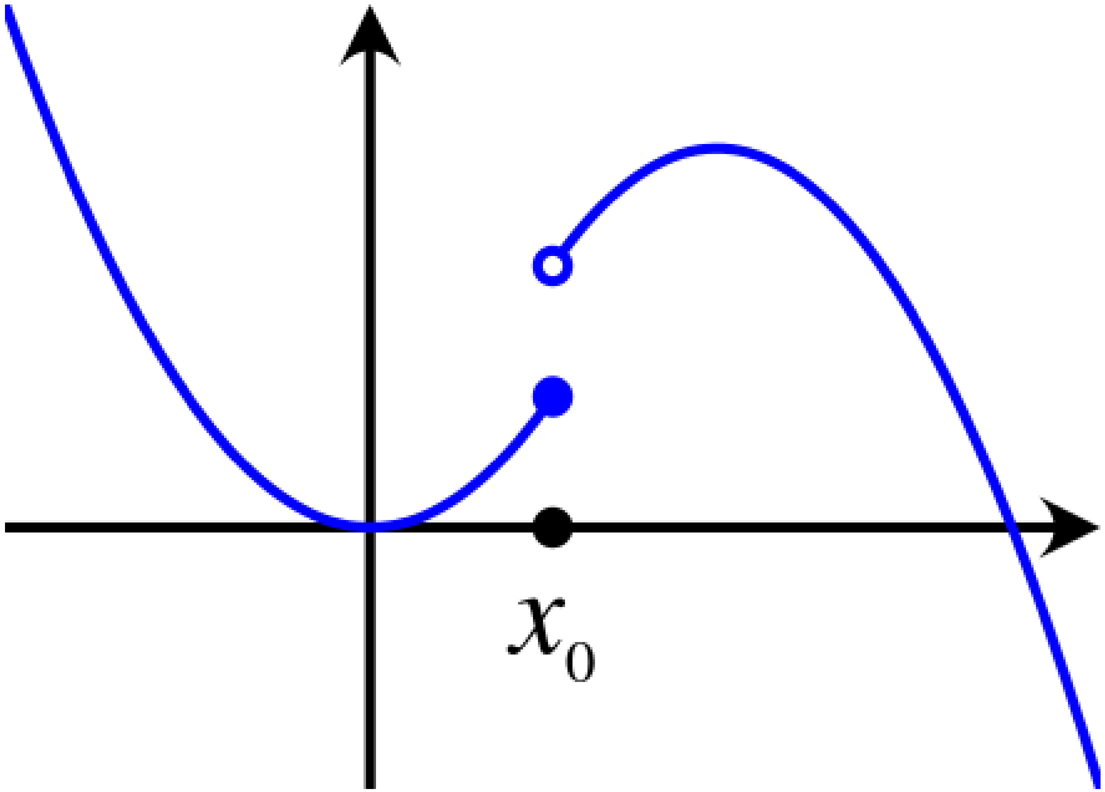
The set $C(\Omega)$ of all continuous real functions defined on a subset Ω of $\mathbb{R}^n$ is a commutative ring with respect to the point-wise defined addition and multiplication of functions.

Is it possible to extend the algebraic operations on $C(\Omega)$ to the set $\mathbb{H}(\Omega)$ in a way which preserves the algebraic structure, that is, the set of $\mathbb{H}(\Omega)$ is a commutative ring with respect to the extended operations?

Upper semi-continuous function (wikipedia)



Lower semi-continuous function (wikipedia)

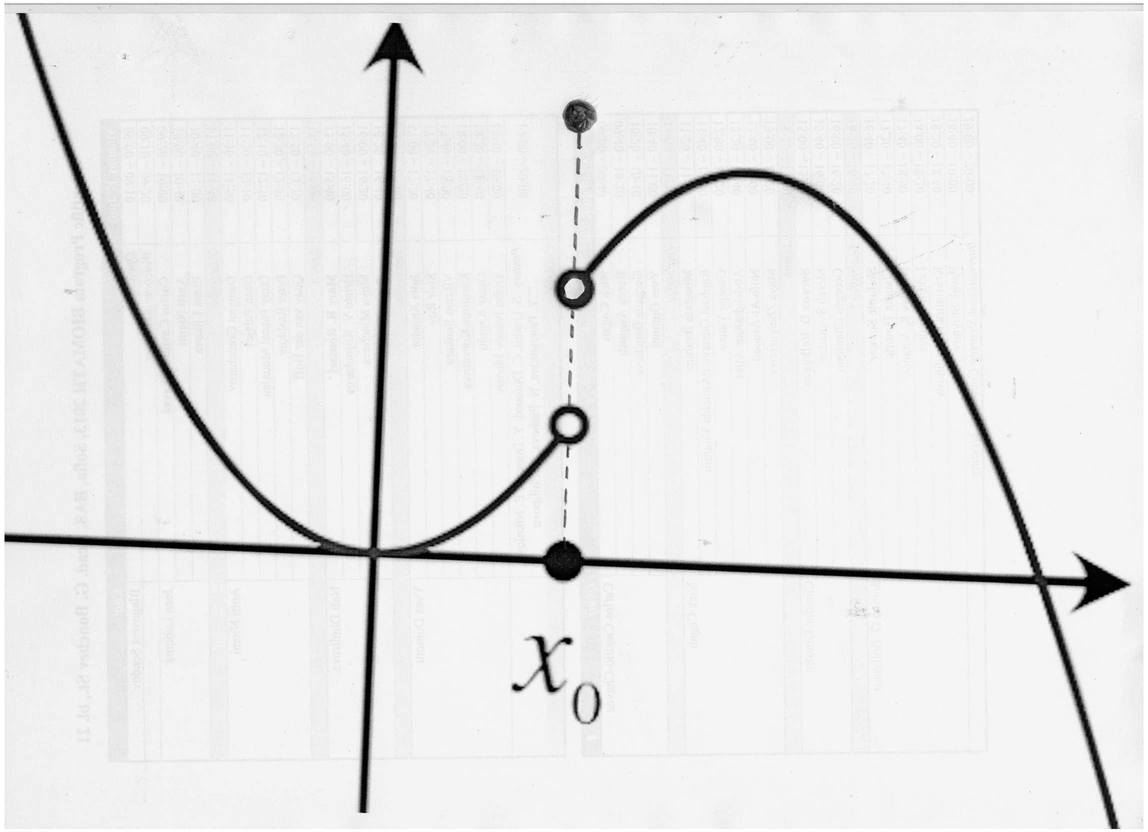


Baire semi-continuous functions: basic definition

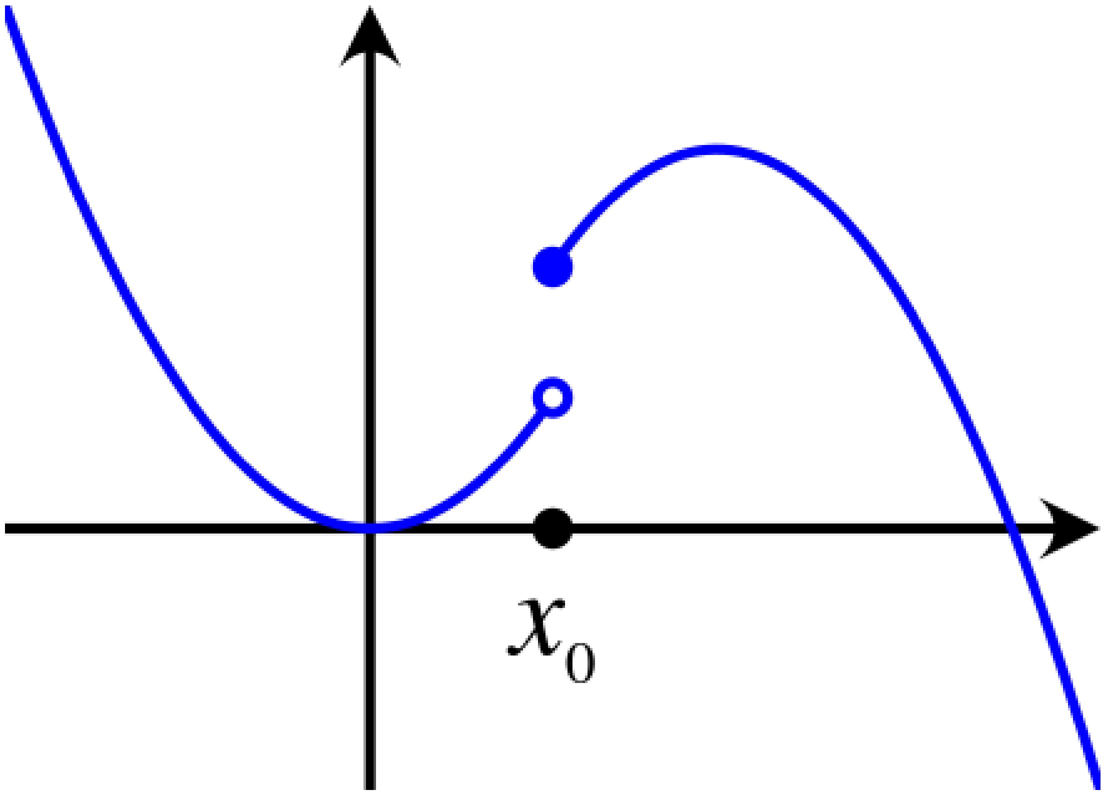
A real-valued function f is upper semi-continuous at a point x_0 if, the function values for arguments near x_0 are either close to $f(x_0)$ or less than $f(x_0)$

A real-valued function f is lower semi-continuous at a point x_0 if, the function values for arguments near x_0 are either close to $f(x_0)$ or greater than $f(x_0)$

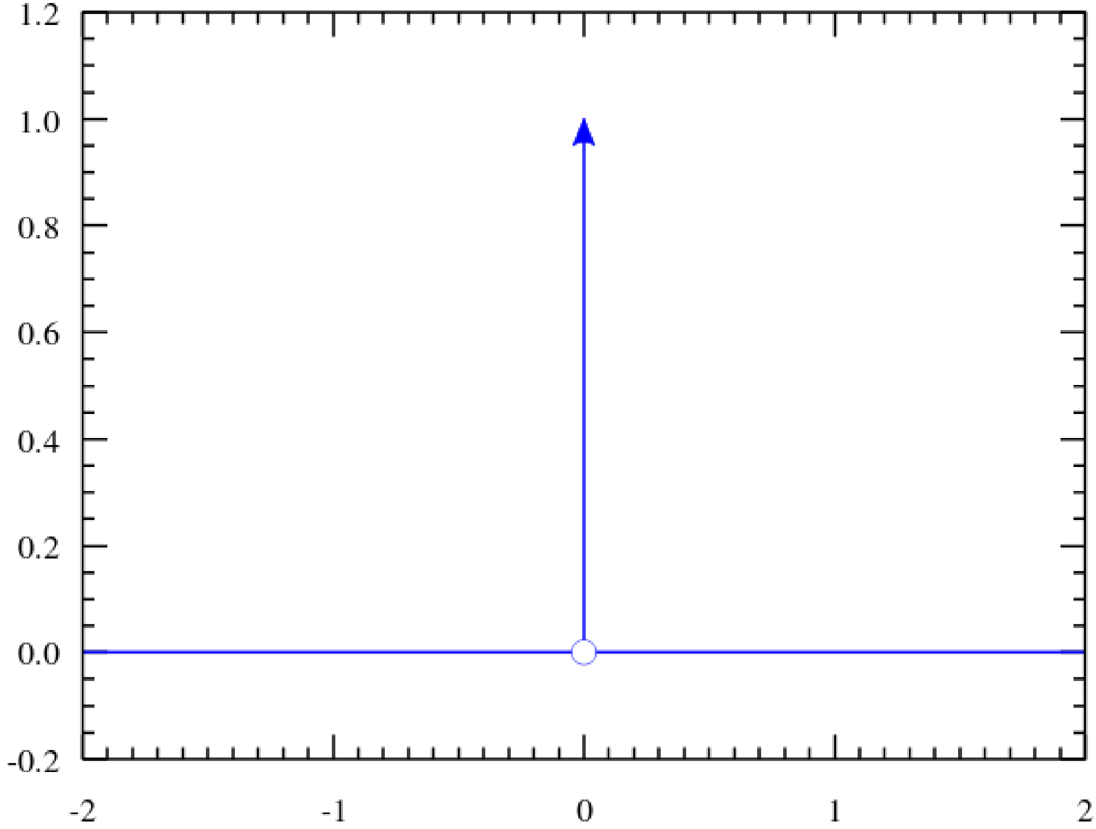
Upper semi-continuous function—Baire operators



Upper normal semi-continuous function—Dilworth operators)



Dirac delta function)



Baire semi-continuous functions: some references

1905 - R. Baire - Lecons sur les Fonctions Discontinues

1950 - R. P. Dilworth - normal upper semicontinuous functions

1953 - A. Horn - normal lower semicontinuous functions

1971 - K. Hardy - operations on normal functions

2005 - 2010 - R. Anguelov, S. Markov - H-continuous interval-valued functions

2009 - 2011 N. Danet - H-continuous interval-valued functions

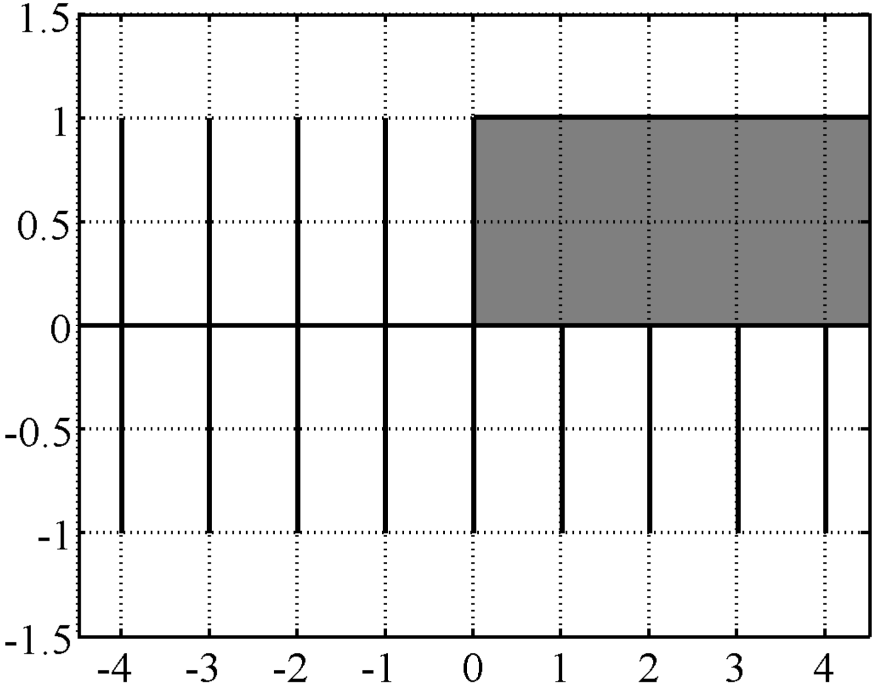
2013 J. H. van der Walt - H-continuous interval-valued functions

The H-continuous functions do not differ to much from the usual real-valued continuous functions because they assume interval values only on a set of first Baire category

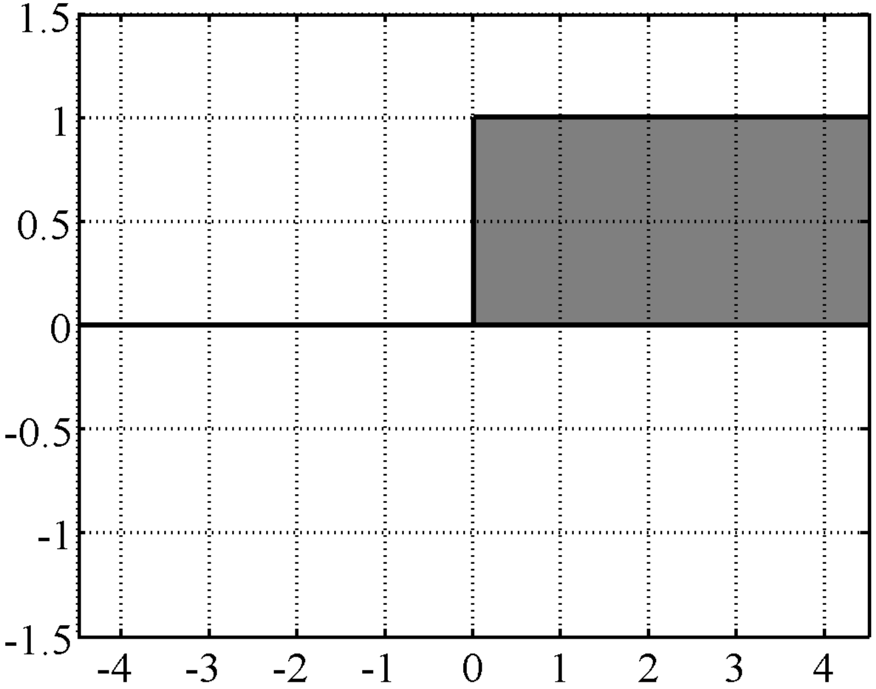
Wikipedia—set of first Baire category:

In topology, a meagre set, also called a set of first Baire category, is a set that, considered as a subset of a (usually larger) topological space, is small or negligible

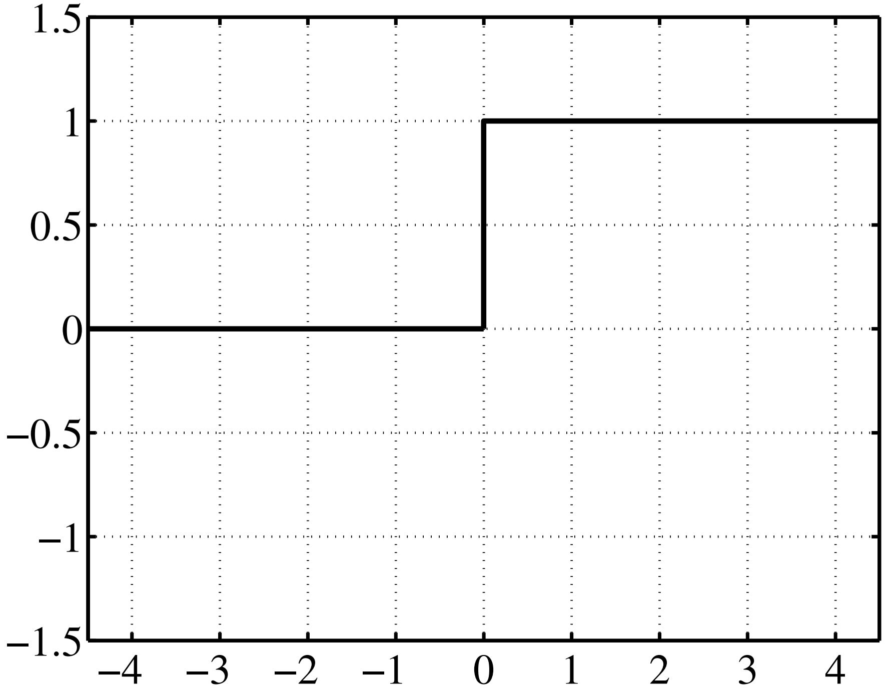
S-continuous interval function



D-continuous interval function



H-continuous interval function



Notation

$\Omega \subseteq \mathbb{R}^n$ open set

$$\begin{aligned} \mathbb{A}(\Omega) &= \{ f : \Omega \to \mathbb{I}\,\mathbb{R}, \ \ f \text{ locally bounded } \} \\ \mathcal{A}(\Omega) &= \{ f : \Omega \to \mathbb{R}, \ \ f \text{ locally bounded } \} \quad \subseteq \quad \mathbb{A}(\Omega) \end{aligned}$$

A real or interval-valued function f on Ω is locally bounded if for every $x \in \Omega$ there exist $\delta > 0$ and $M \in \mathbb{R}$ such that

$$|f(y)| < M, \quad y \in B_\delta(x)$$

$$B_\delta(x) = \{ y \in \Omega : ||x - y|| < \delta \}$$

Baire operators

D is a dense subset of Ω

The lower/upper Baire operators

$I(D,\Omega,\cdot),\ S(D,\Omega,\cdot):\ \mathbb{A}(D) \to \mathcal{A}(\Omega)$

are defined for $f = [\underline{f},\ \overline{f}] \in \mathbb{A}(D)$ and $x \in \Omega$ by

$$I(D,\Omega,f)(x) \quad = \quad \sup_{\delta>0} \inf \ \{\underline{f}(y):\ y \in B_\delta(x) \cap D\}$$

$$S(D,\Omega,f)(x) \quad = \quad \inf_{\delta>0} \sup \ \{\overline{f}(y):\ y \in B_\delta(x) \cap D\}$$

Graph completion operator

$$F:\ \mathbb{A}(D) \rightarrow \mathbb{A}(\Omega)$$

$$F(D,\Omega,f)(x) = [I(D,\Omega,f)(x),\ \ S(D,\Omega,f)(x)],\ \ x \in \Omega,\ \ f \in \mathbb{A}(D)$$

For $D = \Omega$ we write

$$I(f) = I(\Omega,\Omega,f), \quad S(f) = S(\Omega,\Omega,f), \quad F(f) = F(\Omega,\Omega,f)$$

Presentations using end-point functions

$f = [\underline{f},\ \overline{f}] \in \mathbb{A}(\Omega)$

$$I(D,\Omega,f) = I(D,\Omega,\underline{f}), \quad S(D,\Omega,f) = S(D,\Omega,\overline{f})$$

$$F(D,\Omega,f) = [I(D,\Omega,\underline{f}),\ S(D,\Omega,\overline{f})]$$

Continuity Concepts

Definition 1 *A function* $f \in \mathbb{A}(\Omega)$ *is S-continuous, if* $F(f) = f$

Definition 2 *A function* $f \in \mathbb{A}(\Omega)$ *is D-continuous if for every dense subset* D *of* Ω *we have* $F(D,\ \Omega,\ f) = f$

Definition 3 *A function* $f \in \mathbb{A}(\Omega)$ *is H-continuous, if for every function* $g \in \mathbb{A}(\Omega)$ *such that* $g(x) \subseteq f(x), \quad x \in \Omega$*,*
we have $F(g)(x) = f(x), \quad x \in \Omega$

H-continuous functions are “thin”

Theorem. For every $f \in \mathbb{H}(\Omega)$ the set

$$W_f = \{x \in \Omega : \ w(f(x)) > 0\}$$

is of first Baire category

Theorem. For every $f, g \in \mathbb{H}(\Omega)$ the set

$$D_{fg} \quad = \quad \{x \in \Omega : \quad w(f(x)) = w(g(x)) = 0\}$$

is dense in Ω ($\Omega \subseteq \mathbb{R}^n$ is open).

Interval arithmetic

$$a = [\underline{a}, \overline{a}], \quad b = [\underline{b}, \overline{b}] \in \mathbb{I}\,\mathbb{R}$$

$$\begin{aligned} a + b &= \{\alpha + \beta : \alpha \in a,\ \ \beta \in b\} \\ a \times b &= \{\alpha\beta : \alpha \in a,\ \ \beta \in b\} \end{aligned}$$

$$\begin{aligned} [\underline{a}, \overline{a}] + [\underline{b}, \overline{b}] &= [\underline{a} + \underline{b},\ \overline{a} + \overline{b}] \\ [\underline{a}, \overline{a}] \times [\underline{b}, \overline{b}] &= [\min\{\underline{a}\underline{b}, \underline{a}\overline{b}, \overline{a}\underline{b}, \overline{a}\overline{b}\},\ \max\{\underline{a}\underline{b}, \underline{a}\overline{b}, \overline{a}\underline{b}, \overline{a}\overline{b}\}] \end{aligned}$$

Interval arithmetic

$f, g \in \mathbb{A}(\Omega), \quad f = [\underline{f}, \overline{f}], \quad g = [\underline{g}, \overline{g}], \quad x \in \Omega$

$$(f+g)(x) = f(x) + g(x) \ = \ [\underline{f}(x) + \underline{g}(x), \overline{f}(x) + \overline{f}(x)]$$

$$\begin{aligned}(f \times g)(x) &= f(x) \times g(x) \\ &= [\min\{\underline{f}(x)\underline{g}(x), \underline{f}(x)\overline{g}(x), \overline{f}(x)\underline{g}(x), \overline{f}(x)\overline{g}(x)\},\ \max\{\ \ldots\ \}]\end{aligned}$$

Example 1

For $f, g \in \mathbb{H}(\mathbb{R})$ given by

$$f(x) = \begin{cases} 0, & \text{if} \quad x < 0, \\ [0,1], & \text{if} \quad x = 0 \\ 1, & \text{if} \quad x > 0 \end{cases}$$

and $g = (-1) \times f(x) \in \mathbb{H}(\mathbb{R})$ we have

$$(f+g)(x) = f(x) + g(x) = \begin{cases} 0, & \text{if} \quad x < 0 \text{ or } x > 0 \\ [-1,1], & \text{if} \quad x = 0 \end{cases}$$

Note that $f + g \notin \mathbb{H}(\mathbb{R})$.

Defining addition and multiplication via interval operations

Let $f, g \in \mathbb{H}(\Omega)$.

Theorem. (a) There exists a unique function $p \in \mathbb{H}(\Omega)$ such that $p(x) \subseteq (f+g)(x),\ x \in \Omega$.

(b) There exists a unique function $q \in \mathbb{H}(\Omega)$ such that $q(x) \subseteq (f \times g)(x),\ x \in \Omega$.

Definition. (a) $f \oplus g$ is the unique H-continuous function satisfying $(f \oplus g)(x) \subseteq (f+g)(x),\ x \in \Omega$;

(b) $f \otimes g$ is the unique Hausdorff continuous function satisfying $(f \otimes g)(x) \subseteq (f \times g)(x),\ x \in \Omega$.

Revisiting Example 1. For $f \in \mathbb{H}(\mathbb{R})$ given by

$$f(x) = \begin{cases} 0, & \text{if} \quad x < 0 \\ [0,1], & \text{if} \quad x = 0 \\ 1, & \text{if} \quad x > 0 \end{cases}$$

and $g = (-1) \times f(x) \in \mathbb{H}(\mathbb{R})$ we have

$$\begin{aligned} (f+g)(x) &= f(x) + g(x) = \begin{cases} 0, & \text{if} \quad x < 0 \text{ or } x > 0 \\ [-1,1], & \text{if} \quad x = 0 \end{cases} \\ (f \oplus g)(x) &= 0\ , \quad x \in \mathbb{R} \end{aligned}$$

The commutative ring $\mathbb{H}(\Omega)$

Theorem. The set $\mathbb{H}(\Omega)$ is a commutative ring with identity with respect to the operations $\oplus$ and $\otimes$.

The operations $\oplus$ and $\otimes$ are not point-wise in general

At a point where both operands have interval values the value of the sum $\oplus$ or the product $\otimes$ are not determined by the values of the operands only at that point but rather by the values of the operands in a neighborhood of the point.

In the special case when one of the operands is a real (point) valued function the operations $\oplus$ and $\otimes$ coincide with the point-wise operations:

$$
\begin{aligned}
(f \oplus g)(x) &= (f + g)(x) \quad \text{if} \quad w(f(x)) = 0 \text{ or } w(g(x)) = 0 \\
(f \otimes g)(x) &= (f \times g)(x) \quad \text{if} \quad w(f(x)) = 0 \text{ or } w(g(x)) = 0
\end{aligned}
$$

The set of H-continuous functions as a linear space

Multiplication by a scalar is defined as multiplication by a constant function. Since the value of this function is a real number this multiplication coincides with the point-wise multiplication

$$(\alpha * f)(x) = \alpha * f(x) = \begin{cases} [\alpha \underline{f}(x), \alpha \overline{f}(x)] & \text{if} \quad \alpha \geq 0 \\ [\alpha \overline{f}(x), \alpha \underline{f}(x)] & \text{if} \quad \alpha < 0 \end{cases}$$

The set $\mathbb{H}(\Omega)$ is a linear space with respect to "$\oplus$" and "$*$"

Can the operations be extended further?

Let $\mathbb{G}(\Omega)$ be the set of all D-continuous interval functions.

Theorem. Assume that the set $\mathcal{P} \subseteq \mathbb{G}(\Omega)$ is closed under inclusion in the sense that

$$\left. \begin{array}{c} f \in \mathcal{P}, \quad g \in \mathbb{G}(\Omega) \\ g(x) \subseteq f(x), \quad x \in \Omega \end{array} \right\} \Longrightarrow g \in \mathcal{P}$$

If $\mathcal{P} \subseteq \mathbb{G}(\Omega)$ is a linear space, then $\mathcal{P} \subseteq \mathbb{H}(\Omega)$

Hence the operations $\oplus$, $\otimes$ cannot be extended further than $\mathbb{H}(\Omega)$ in a way preserving the algebraic structure of $C(\Omega)$.

Some open problems

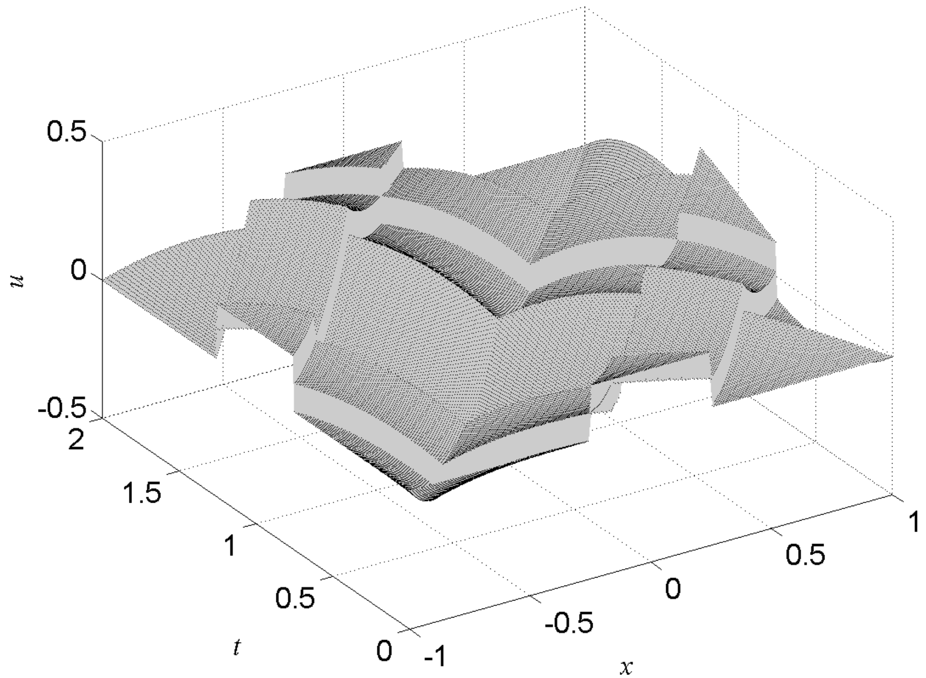
To further develop numerical computations with H-cont. fns by considering finite dimensional spaces induces by various basis

To consider quasivector spaces of D-cont. fncs (by embedding the additive semigroup in a group)

To further explore applications to solution of PDE problems, such as the viscosity solution for the Hamilton-Jacobi equation

illustration on the next slide

2D case—viscosity solution for the Hamilton-Jacobi PDE



Related problems from enzyme kinetics and cell growth dynamics

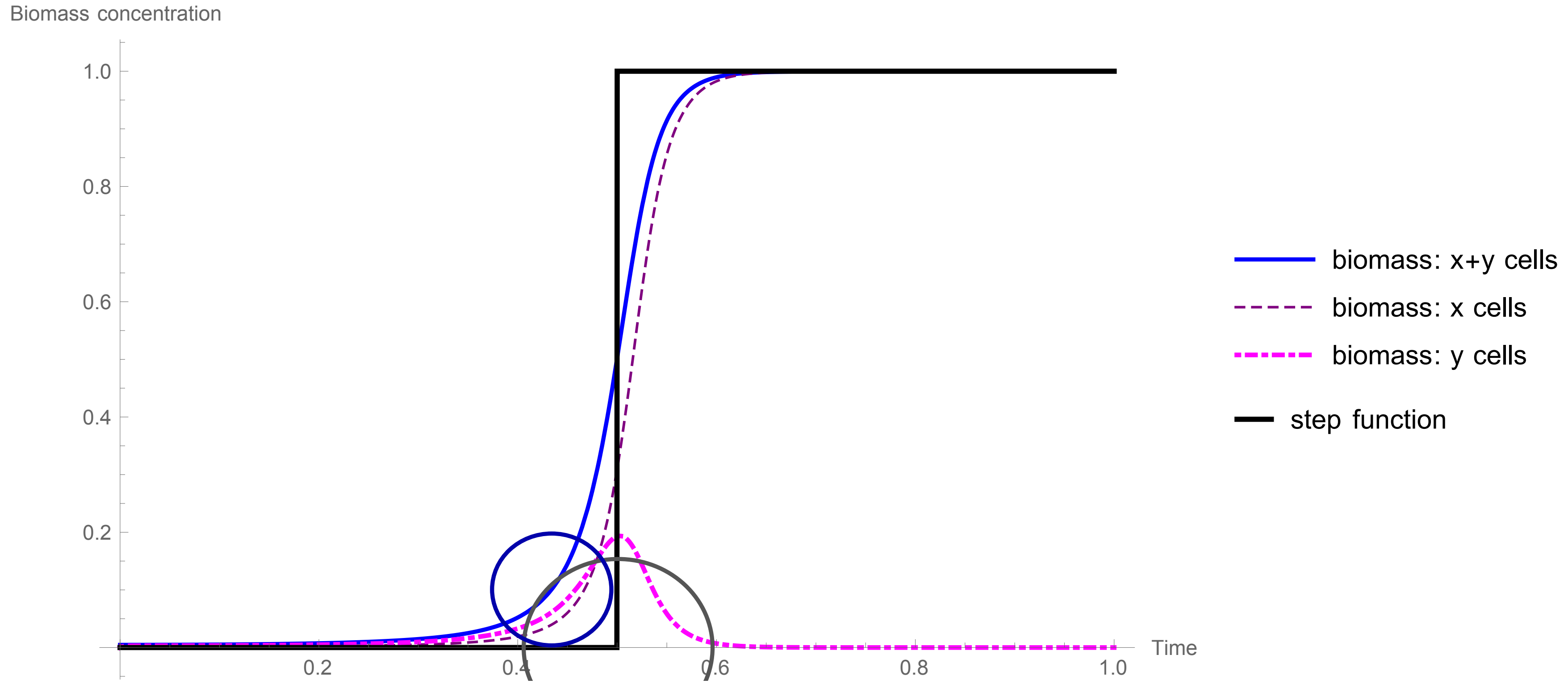
To approximate in Hausdorff metric the interval Dirac function by means of the c-solution of Henri enzyme kinetic system (c is the enzyme complex concentration)

To approximate in Hausdorff metric the interval Heaviside step function by means of the biomass solution of the basic 2-state cell growth model

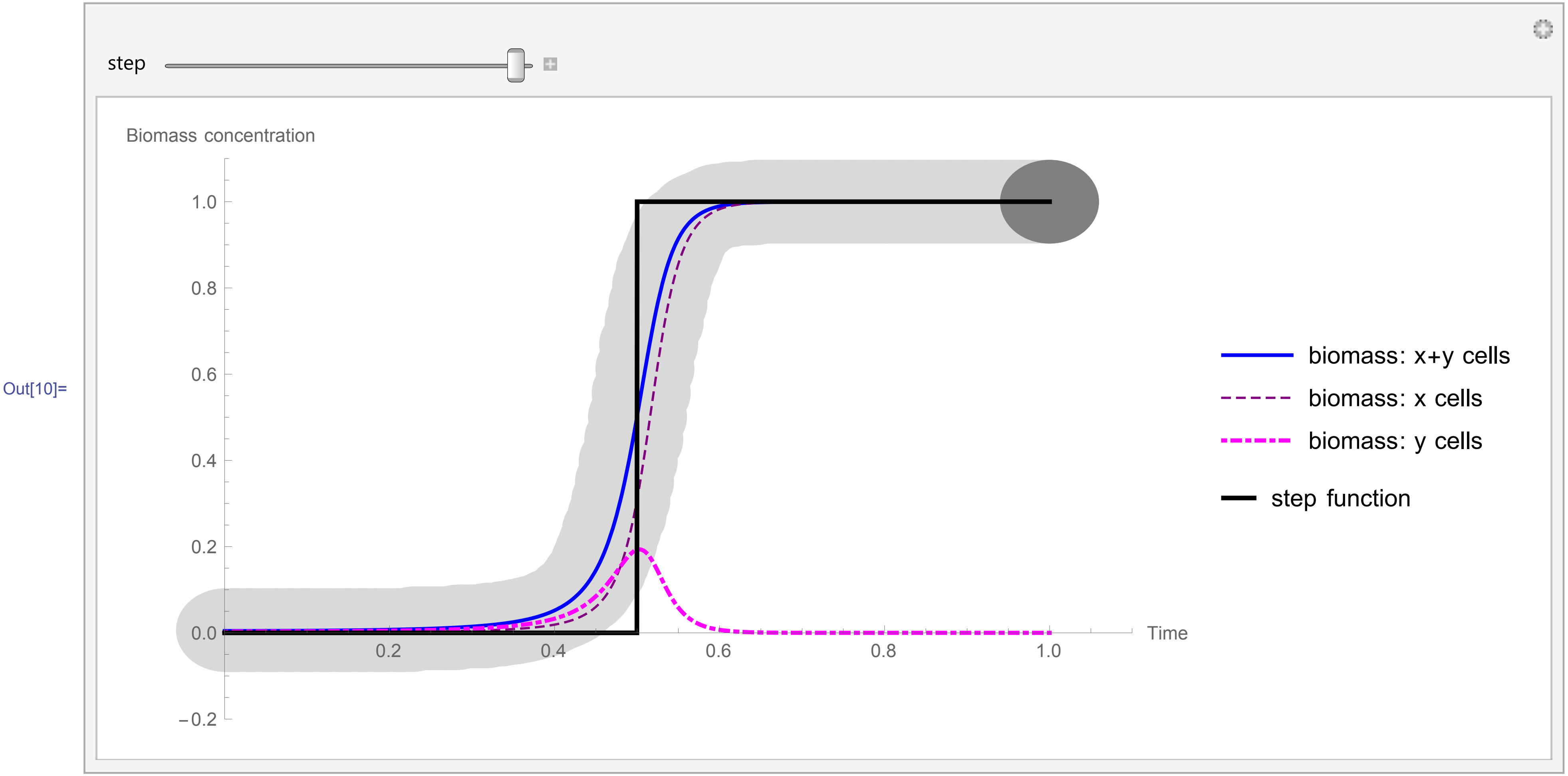
Illustrated on the next slides

H-distance

Out[9]=



H-distance


step
Biomass concentration
1.0
0.8
0.6
0.4
0.2
0.0
−0.2
0.2
0.4
0.6
0.8
1.0
Time
biomass: x+y cells
biomass: x cells
biomass: y cells
step function
Out[10]=

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