On Unsolvability of Overdetermined Interval Linear Systems

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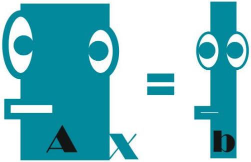
Introduction



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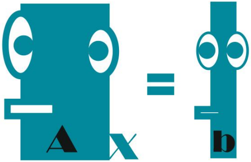

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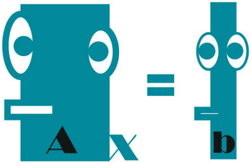

- Let $\boldsymbol{A}x = \boldsymbol{b}$ be an interval linear system.
- We call it **overdetermined** system, if it has more equations than variables.

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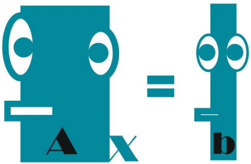

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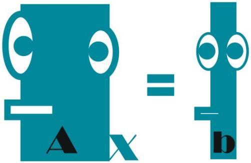

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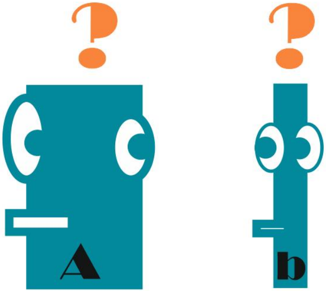
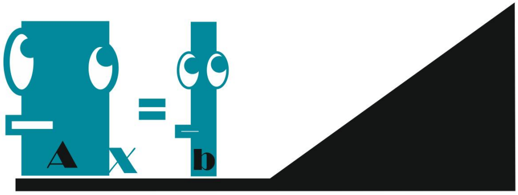
- An interval system is called **unsolvable** , when its Σ set is empty. Otherwise it is called **solvable** .

Solvability and Unsolvability



Complexity



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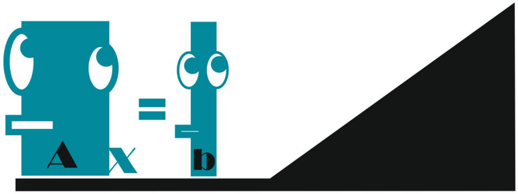

- Computing the set Σ is NP-hard

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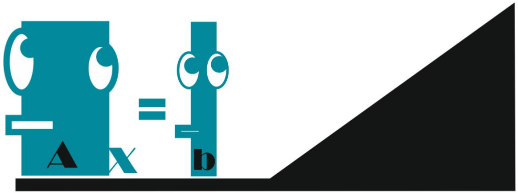

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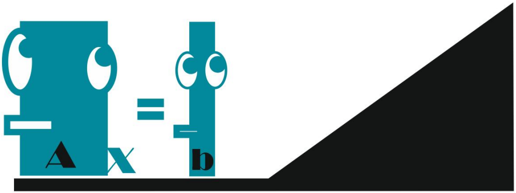

- Computing the set Σ is NP-hard
- Computing exact bounds on Σ is NP-hard
- Deciding whether Σ is nonempty is NP-hard

Existing results

- There are many existing results on unsolvability and solvability

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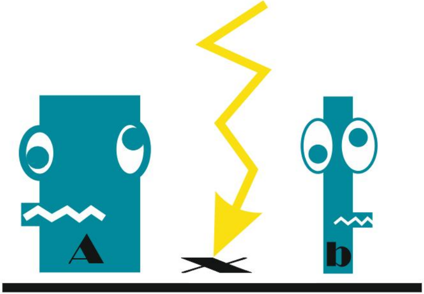
- We would like to introduce some simple practically usable
- We mostly focus on overdetermined interval systems

Unsolvability



Overview

- Gaussian elimination
- Linear programming
- Subsquares
- Full colum rank
- LSQ

1. Gaussian elimination

- GE for overdetermined ILS was introduced by Hansen (2006)

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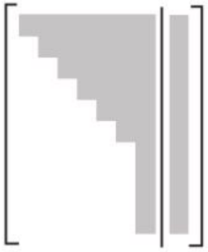
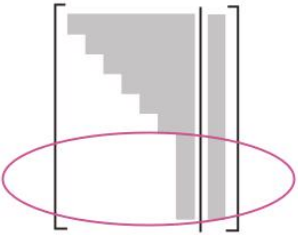
- We simply eliminate elements (using interval operations) under the main diagonal to the following shape



1. Gaussian elimination

- Let us take a look at the last $(m - n + 1)$ equations



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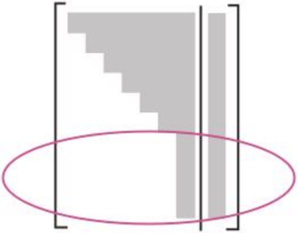

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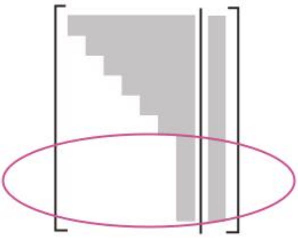

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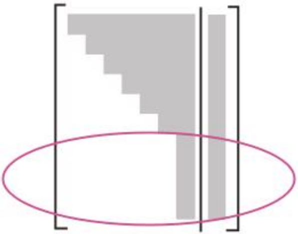

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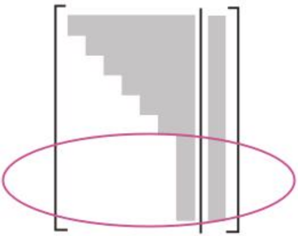

- Each of them is of shape $x_n = [\,\underline{\cdot}\,,\,\overline{\cdot}\,]$
- If intersection of these intervals is empty, then the system is unsolvable
- Only for small systems
- We cannot use preconditioning

2. Linear programming

Oettli-Prager theorem

Vector $x \in \mathbb{R}^n$ is a solution of an interval system if and only if

$$|A_c x - b_c| \leq A_\Delta |x| + b_\Delta.$$

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- We get a system of linear inequalities in each orthant

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- Verified linear programming can return "I don’t know" answer!

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- Idea: We select some subsystems and compute their enclosures
- If the intersection of those enclosures is empty the overdetermined system is unsolvable

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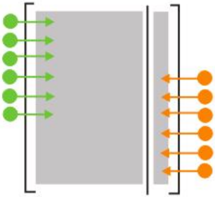

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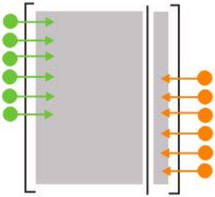

- Works fine even for large intervals (for small intervals usually 2 subsystems are enough)

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- Interval matrix $\boldsymbol{A}$ has full column rank if every $A \in \boldsymbol{A}$ has full column rank
- If extended matrix of an interval system $[\boldsymbol{A}|\boldsymbol{b}]$ has full column rank, then the ILS is not solvable

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- $\Rightarrow$ if $||I - R\boldsymbol{A}|| < 1$ then $\boldsymbol{A}$ has full column rank
- Good choice is $R \approx A_c^+$

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- $\Sigma \subseteq \Sigma_{lsq}$
- We are able to get an interval enclosure $\boldsymbol{x}$ of Σ_{lsq} (*verifylss*)

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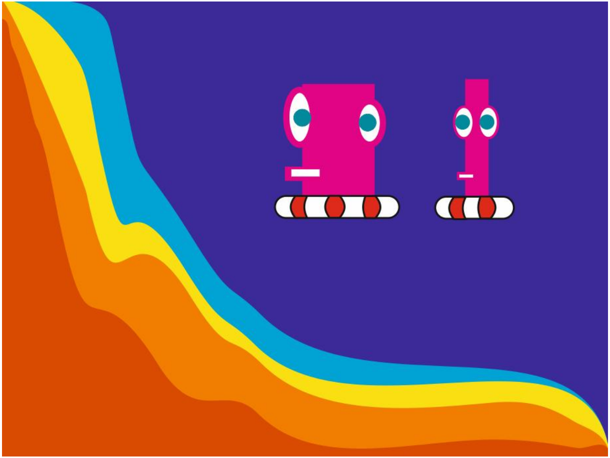
- Let us compute $\boldsymbol{A}\boldsymbol{x} - \boldsymbol{b}$ using interval operations
- If this interval vector still does not contain zero vector, then for no $Ax = b$ exists y such that $Ay - b = 0$
- Therefore, the interval system is unsolvable

Testing



Visualisations of testing 1

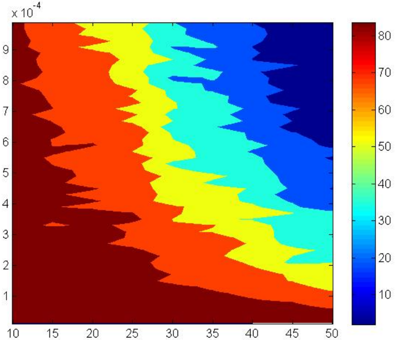


Figure: Least squares

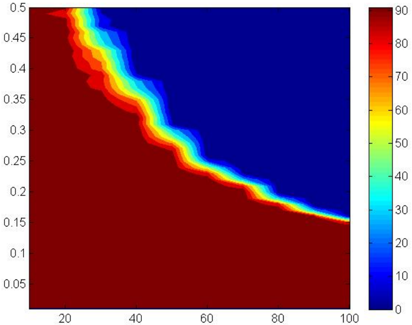


Figure: Full colum rank

Visualisations of testing 2

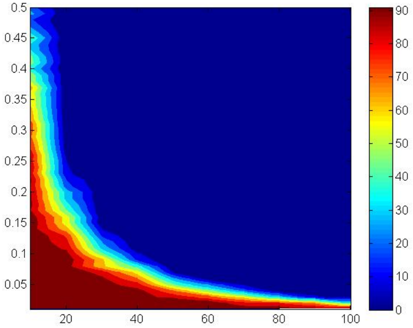
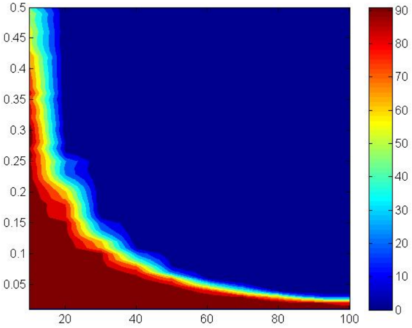


Figure: Subsquares 5 sys.



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  - Verified solution enclosure computation
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  - Parallelization
- New conditions in progress

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best_final_image.jpg