Towards bounds on the radius of nonsingularity

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Radius of nonsingularity

Original task

- For given square matrix
  measure the distance to the "nearest" singular one

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Definition (Radius of nonsingularity)

Given a matrix $A \in \mathbb{R}^{n\times n}$, the *radius of nonsingularity* is defined by

$$d(A) := \inf \{\varepsilon > 0;\ \exists \text{ singular } B : |a_{ij} - b_{ij}| \leq \varepsilon\ \forall i, j\}.$$

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Note

- In fact, a lot of results are for generalized version

$$d(A, \Delta) := \inf\{\varepsilon > 0;\ \exists \text{ singular } B : |a_{ij} - b_{ij}| \leq \varepsilon \Delta_{i,j}\ \forall i, j\}.$$

where $\Delta \in \mathbb{R}^{n \times n}$ is a given non-negative matrix

Motivation to determine radius of nonsingularity

Data uncertainty

- Entries of matrix $A = (a_{i,j})$ represents results of experiment
- Measurement device ensures some (uniform) precision δ
- It is thus guaranteed that the (unknown) actual values is in

$$[a_{i,j} - \delta, a_{i,j} + \delta]$$

- Question: Is A suitable for further processing?
  - Yes: $\delta < d(A)$
  - No: $\delta \geq d(A)$

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Rounding fixed-point arithmetic

- Assume A having irrational entries
  e.g. formal derivation of data like distance may end in $\sqrt{2}$
- Rounding before numerics $\rightarrow \tilde{A}$
- $\|A - \tilde{A}\| > d(\tilde{A}) \quad \Rightarrow$ problems
  presence of singular matrix within precision $\Rightarrow \tilde{A}$ does not reflect properties of A

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Computing radius of nonsingularity

Proposition (Poljak and Rohn, 1993))

For each non-singular A there hols

$$d(A) := \frac{1}{\|A^{-1}\|_{\infty,1}}$$

- where $\|\cdot\|_{\infty,1}$ is a matrix norm defined as

$$\|M\|_{\infty,1} := \max\{\|Mx\|_1;\ \|x\|_\infty = 1\} = \max\{\|Mz\|_1;\ z \in \{\pm 1\}^n\}$$

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- Computing exact $\|A\|_{\infty,1}$ can be reduced to

  max-cut of weighted graph
- This provides arguments for (Poljak and Rohn, 1993):

  Computing $d(A)$ is NP-hard

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- There is no polynomial time algorithm for approximating $d(A)$ with a relative error at most $\frac{1}{4n^2}$ (Rohn, 1996)

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Rohn, J. (1996). Technical Report 686, ICS Prague.

Bounds on radius of nonsingularity

There are several bounds for $d(A)$

- Rohn provides (Rohn, 1996)

$$\frac{1}{\rho(|A^{-1}|E)} \leq d(A) \leq \frac{1}{\max\limits_{i=1,\dots,n} (E|A^{-1}|)_{ii}}$$

- Rump (Rump, 1997 and 1997b) developed the estimations

$$\frac{1}{\rho(|A^{-1}|E)} \leq d(A) \leq \frac{6n}{\rho(|A^{-1}|E)}.$$

Rohn, J. (1996): *Technical Report* 686, ICS Prague.
Rump, S. M. (1997): *Linear Multilinear Algebra*, 42(2):93–107.
Rump,S. M. (1997b): *SIAM J. Matrix Anal. Appl.*, 18(1):83–103.

Goal and main idea

Overall goal is

- Provide (randomized) aprroximate algorithm for $d(A)$
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- Approach

  1. Develop approximation of $\|M\|_{\infty,1}$ using semi-definite relation (This gives randomized approximation algorithm)
  2. This provides also approximation for $d(A)$ through

$$d(A) := \frac{1}{\|A^{-1}\|_{\infty,1}}$$

SDP relaxation

The problem of computing $\|M\|_{\infty,1}$ can be formulated as

$$\max \sum_{i,j=1}^{n} m_{ij} x_i y_j \quad \text{subject to} \quad x, y \in \{\pm 1\}^n \tag{1}$$

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The SDP relaxation

- replacing **discrete variables** $x_i, y_j \in \{\pm 1\}$, $i, j = 1, \ldots, n$ by **unit vectors** $u_i, v_j \in \mathbb{R}^n$, $i, j = 1, \ldots, n$ as follows

$$\max \sum_{i,j=1}^{n} m_{ij} u_i^T v_j \quad \text{subject to} \quad u_i, v_i \in \mathbb{R}^n, \tag{2}$$

$$\text{where } \|u_i\|_2 = \|v_i\|_2 = 1, \;\; i = 1, \ldots, n$$

Gärtner, B. and Matoušek, J (2012): Springer

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Connection

- For any feasible solution x, y to (1) $\rightsquigarrow$ solution u, v to (2) as
  - $u_i := (0, \dots, 0, x_i)$ and $v_i := (0, \dots, 0, y_i)$, $i = 1, \dots, n$
- Thus (2) is relaxation of (1)
  - the OV to (2) is an upper bound on the OV to (1)

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Further adjustment

Starting at transformed problem

$$\max \sum_{i,j=1}^{n} m_{ij} u_i^T v_j \ \text{ subject to } \ u_i, v_i \in \mathbb{R}^n,$$

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Define the matrix $Z \in \mathbb{R}^{2n \times 2n}$ as $Z := U^T U$

- so that U has the columns $u_1, \dots, u_n, v_1, \dots, v_n$
- note also that Z is positive semidefinite (denoted by $Z \succeq 0$)

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We can thus transform above program into

$$\max \sum_{i,j=1}^{n} m_{ij} z_{i,j+n} \quad \text{subject to} \quad Z \succeq 0, z_{ii} = 1, \; i = 1, \ldots, 2n \qquad (3)$$

Solving the problem

The semidefinite problem

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- given by (Grötschel, Lovász, Schrijver, 1981)
- for optimal value γ holds $\gamma \geq \|M\|_{\infty,1} - \varepsilon$ and thus

$$\gamma + \varepsilon \geq \|M\|_{\infty,1}$$

  is a bound for our norm

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Handling

- Utilize optimal solution of (3) resp. (2) to find optimal solution to (1)

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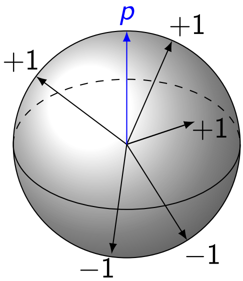
Optimal solution of transformed problem

Rounding the vector solution

- Let approximate optimal solution of transformed problem be $u_1^*, \ldots, u_n^*, v_1^*, \ldots, v_n^*$
- Use $p \in \mathbb{R}^n$ being a unit vector via following mapping

$$w \mapsto \begin{cases} 1 & \text{if } p^T w \geq 0, \\ -1 & \text{otherwise} \end{cases} \tag{4}$$

to determine solution to original problem



Randomized algorithm computing $d(A)$

1. Determine inverse matrix $M = A^{-1}$
2. Compute approximation of optimal solution of
$$\max \sum_{i,j=1}^{n} m_{ij} u_i^T v_j \quad \text{subject to} \quad u_i, v_i \in \mathbb{R}^n, \tag{2}$$
3. Choose randomly unit vector $p \in \mathbb{R}^n$ and apply
$$w \mapsto \begin{cases} 1 & \text{if } p^T w \geq 0, \\ -1 & \text{otherwise} \end{cases} \tag{4}$$
which provides solution to problem
$$\|M\|_{\infty,1} = \max \sum_{i,j=1}^{n} m_{ij} x_i y_j \quad \text{subject to} \quad x, y \in \{\pm 1\}^n \tag{1}$$
4. Finally compute
$$d(A) := \frac{1}{\|A^{-1}\|_{\infty,1}} = \frac{1}{\|M\|_{\infty,1}}$$

Expected value of the solution

Lemma (Gärtner and Matoušek, 2012)

Let $u, v \in \mathbb{R}^n$ be unit vectors. The probability that the mapping (4) maps u and v to different values is $\frac{1}{\pi} \arccos u^T v$.

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Apply this lemma

- Let have a feasible solution to (1)
  $x_1^*, \ldots, x_n^*, y_1^*, \ldots, y_n^*$
  determined by images (of defined mapping)
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  determined by images (of defined mapping)
  $u_1^*, \ldots, u_n^*, v_1^*, \ldots, v_n^*$
- By the Lemma, the expected value of the solution is

$$\sum_{i,j} m_{ij} \left(1 - \frac{1}{\pi} \arccos {u_i^*}^T v_j^*\right) - m_{ij} \frac{1}{\pi} \arccos {u_i^*}^T v_j^*$$

$$= \sum_{i,j} m_{ij} \left(1 - \frac{2}{\pi} \arccos {u_i^*}^T v_j^*\right).$$

Bound of the expected value of the solution

Looking for bound of expected value

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Goemans-Williamson ratio α for approximating MAX-CUT

- It value is $\alpha \approx 0.87856723$
- It represents optimal value of the problem $\min_{z\in[-1,1]} \frac{2\arccos z}{\pi(1-z)}$

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Using this constant we can bound the term from expected value

Lemma

For each $z \in [-1,1]$ we have

$$\alpha z + \alpha - 1 \leq 1 - \frac{2}{\pi}\arccos z \leq \alpha z + 1 - \alpha.$$

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Applying the bound

Now, we use the Lemma to find a lower bound to expected value

- Let $i, j \in \{1, 2, \ldots, n\}$, then

$$m_{ij}\left(1 - \frac{2}{\pi}\arccos z\right) \geq \begin{cases} m_{ij}(\alpha z + \alpha - 1) & \text{if } m_{ij} \geq 0, \\ m_{ij}(\alpha z - \alpha + 1) & \text{otherwise,} \end{cases} \quad \text{or}$$

$$m_{ij}\left(1 - \frac{2}{\pi}\arccos z\right) \geq m_{ij}\alpha z + |m_{ij}|(\alpha - 1).$$

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- Thus the lower bound to expected value is

$$\sum_{i,j} m_{ij}\left(1 - \frac{2}{\pi}\arccos {u_i^*}^T v_j^*\right) \geq \sum_{i,j} m_{ij}\alpha {u_i^*}^T v_j^* + |m_{ij}|(\alpha - 1)$$

$$= \alpha\gamma + (\alpha - 1)e^T|M|e,$$

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Hence we have an expected lower bound on $\|M\|_{\infty,1}$

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Determining approximation ratio

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The right-hand side depends on the entries of M

- we can employ the estimate $\|M\|_{\infty,1} \leq e^T|M|e$ to obtain

  $$\|M\|_{\infty,1} \geq \alpha\gamma + (\alpha - 1)e^T|M|e \geq \alpha\gamma + (\alpha - 1)\|M\|_{\infty,1},$$

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  $$\|M\|_{\infty,1} \geq \frac{\alpha}{2-\alpha}\gamma.$$

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This gives us a randomized algorithm with the approximation ratio

$$\frac{\alpha}{2-\alpha} \approx 0.78343281$$

Conclusions

- Computing radius of nonsingularity
  - Randomized approximation of with a constant relative error

$$\approx 0.78343281$$

  - Using relaxation to SDP and principles from approximating MAX-CUT
- Perspectives
  - Derandomization of the algorithm

Thank you


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