Verified localization of trajectories with prescribed behaviour in the forced damped pendulum

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Definition of chaos (not complete)

- Consider two areas in the plane (L and R), and a continuous map (φ) from plane to plane.
- If one point is in L (or R), then we say it is "L" (or "R")
- The notation for bi-infinite L/R sequence is $..., e_{-1}, e_0, e_1, ...$ where $e_i \in \{L, R\}$, for all i.
- If we can give a point (p) in a region for all bi-infinite L/R sequences for which

$$..., \varphi^{-1}(p) \in e_{-1}, \varphi^{0}(p) \in e_0, \varphi^{1}(p) \in e_1, ...,$$

then we say that the system is chaotic in that region.

A method to prove the chaotic behavior

- Find such zones (a, b, c, and d), for which the images are located as on the figure (horseshoe method).
- In this case, there exist points for all bi-infinite series. This is a result from the Miranda theorem.

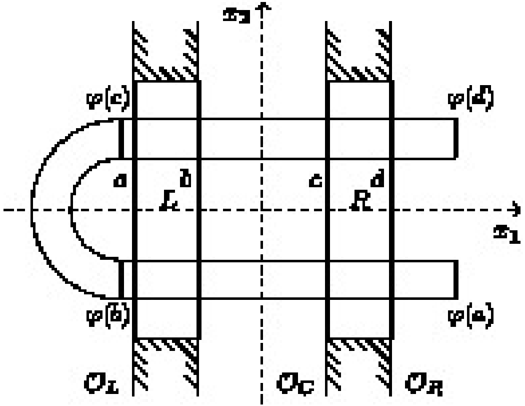


Figure: A classical Smale horseshoe

Other horseshoe

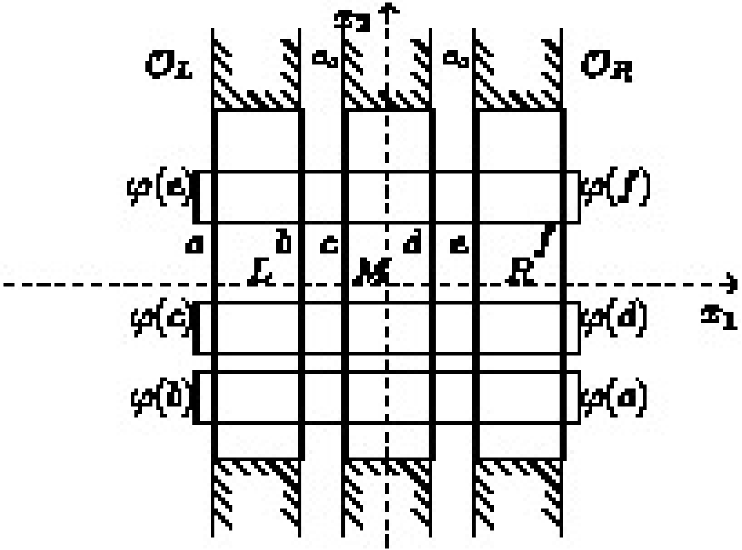


Figure: A Smale horseshoe for Σ^3 chaos

The forced damped pendulum

Consider a forced damped pendulum with these parameters:

- The mass and the length of the pendulum are both unit.
- The friction factor is $b = 0.1$. The friction depends on the speed of the pendulum.
- The degree of forcing is $\cos(t)$, where t is the time.
- The differential equation:

$$x'' = \cos(t) - 0.1x' - \sin(x),$$

where x is an angle of the pendulum, and x' is speed of the pendulum.

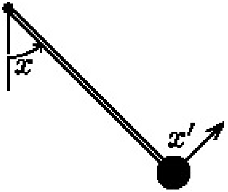


Figure: The forced damped pendulum

The forced damped pendulum (movie)

(Loading the Movie)

Notations

Let an I_k be a time interval: $[2k\pi, 2(k+1)\pi]$. Let us consider those trajectories, for which one of the following happens during the I_k time interval:

- the pendulum goes clockwise through the bottom position exactly once ($\epsilon_k = \ominus$),
- the pendulum does not go through the bottom position ($\epsilon_k = \otimes$), or
- the pendulum goes counterclockwise through exactly once ($\epsilon_k = \oplus$).

We do not consider those trajectories where the pendulum does something else.

Theorem

Theorem

Given any bi-infinite sequence of events : $\ldots, \epsilon_{-1}; \epsilon_0; \epsilon_1; \ldots$ *with* $\epsilon_k \in \{\ominus, \otimes, \oplus\}$, *there exists a solution of our differential equation that during each time interval* $[2k\pi; 2(k+1)\pi)]$ *will do* ϵ_k.[1]

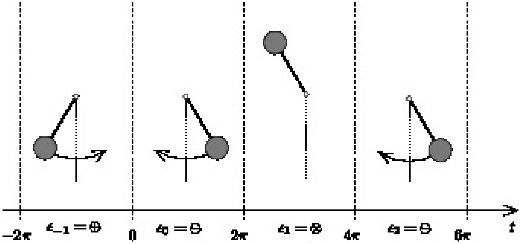


Figure: Four length example for Theorem

[1] Hubbard, John H.: The Forced Damped Pendulum: Chaos, Complication and Control. American Mathematical Monthly, 8:741-758, 1999.

Frame of proof

If we want to prove it, we must solve two subproblems:

- to show a Smale horseshoe between two Poincaré sections, and

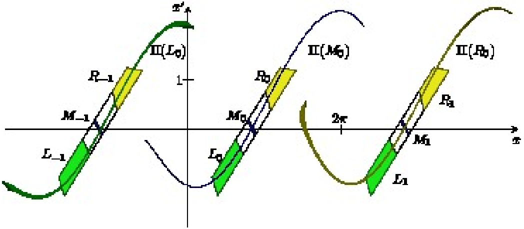


Figure: Illustration of covering

- to show what the pendulum does during I_k.

Remark: The angle of the pendulum is 2π periodic, so it is enough to analyze only one period of the pendulum to prove the existence of a Horseshoe.

The problem of the trajectories verified location

- We present a fitting verified numerical technique capable to find long trajectory segments with prescribed qualitative behaviour and thus shadowing different types of chaotic trajectories with large (theoretically, with arbitrary) precision.
- For example, we can achieve that our pendulum goes through any specified finite sequence of gyrations by choosing the initial conditions correctly.

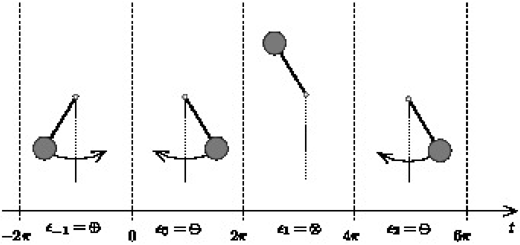


Figure: Four length example for Theorem

Objective function

- The search for a starting point for the expected series was modelled as a constrained global optimization problem.
- We add a nonnegative value proportional to how much the given condition was hurt, plus a fixed penalty term in case at least one of the properties was not satisfied.
- In the objective function we used the Hausdorff distance of the aimed region of the pendulum angle and speed (E), and the union of inclusions boxes of trajectories I, which is a series of rectangle shaped, two dimensional regions, each one of them contains a part of the whole trajectories:

$$\max_{z \in I} \inf_{y \in E} d(z, y),$$

where $d(z, y)$ is a given metric, a distance between two two-dimensional points.

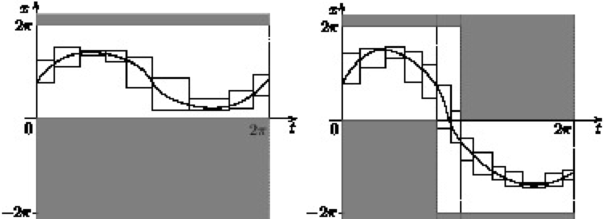
Expected regions

- The aimed region of $\otimes$, is

$$E_{\otimes} = \left\{ (x, x'),\ \text{where } 0 < x < 2\pi \right\}.$$

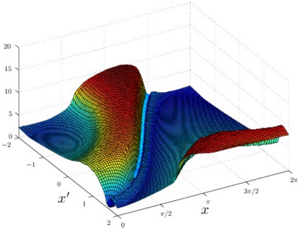
- The expected region of $\ominus$ is

$$E_{\ominus} = \left\{ \begin{array}{l} (x, x'),\ \text{where } 0 < x < 2\pi, \\ \text{before the intersection} \end{array} \right\},$$
$$\cup \left\{ \begin{array}{l} (x, x'),\ \text{where } -2\pi < x < 2\pi \text{ and } x' < 0, \\ \text{during the intersection} \end{array} \right\},$$
$$\cup \left\{ \begin{array}{l} (x, x'),\ \text{where } -2\pi < x < 0, \\ \text{after the intersection} \end{array} \right\}.$$

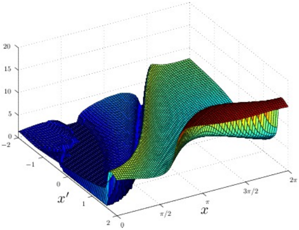


Objective functions

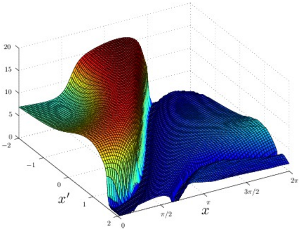
- Objective functions for trajectories of unit length.



(c) The $\otimes$ case.



(d) The $\oplus$ case.



(e) The $\ominus$ case.

Found starting points for the length three series (examples).

S	X	ZO	FE	CT [2]
ˇ⊕⊕⊕	(3.5145566; 1.1854134)	3	2 305	666
ˇ⊕⊕⊗	(3.541253; 1.1780008)	1	3 356	965
ˇ⊕⊕⊖	(4.1354217; 1.1146838)	9	1 489	431
ˇ⊗⊕⊕	(2.6045829; 0.056101674)	2	3 680	1 061
ˇ⊗⊕⊗	(2.6558599; 0.004679824)	1	11 882	3 439
ˇ⊗⊕⊖	(2.5851486; 0.081902247)	6	2 054	594
ˇ⊗⊗⊕	(2.6840309; −0.024118557)	1	8 940	2 582
ˇ⊗⊗⊗	–	0	8 885	2 573
ˇ⊗⊗⊖	(2.4871575; 0.17213042)	1	2 782	803

[2] Here S stands for series, X for the suitable point, ZO for the number of different optimization points with zero objective function value, FE for the number of function evaluations, and CT for CPU time in seconds.

The found starting points for different length series.

L	S	X	ZO [3]
1	$[2._4^8; 0._0^2]$	$(2.7108515; -0.030099507)$	12
2	$[2._6^7; 0._0^1]$	$(2.6469962; 0.013297356)$	4
3	$[2.63_4^5; 0.02_6^7]$	$(2.6342106; 0.026105974)$	1
4	$[2.634_2^3; 0.026_0^1]$	$(2.634273; 0.026043388)$	1
5	$[2.63427_2^4; 0.02604_2^4]$	$(2.6342733; 0.026043083)$	0

[3] Here L stands for the series length, S for the search area (Here $2._4^8$ means the $[2.4, 2.8]$ interval.), X for the suitable point, ZO for the number of zero optimum values.

References I

J.H. Hubbard:
The Forced Damped Pendulum: Chaos, Complication and Control.,
American Mathematical Monthly, 1999

N. Nedialkov:
VNODE, A Validated Solver for Initial Value Problems for Ordinary Differential Equations.,
www.cas.mcmaster.ca/∼nedialk

T. Csendes, B. Bánhelyi and L. Hatvani:
Towards a computer-assisted proof for chaos in a forced damped pendulum equation,
Journal of Computational and Applied Mathematics, **199**:378–383 (2007)

B. Bánhelyi, T. Csendes, B.M. Garay and L. Hatvani:
A computer-assisted proof for Σ_3-chaos in the forced damped pendulum equation,
SIAM Journal on Applied Dynamical Systems, **7**:843-867, 2008 (2008)

B. Lévai, and B. Bánhelyi:
An optimization technique for verified location of trajectories with prescribed geometrical behaviour in the chaotic forced damped pendulum,
Central European Journal of Operations Research, submitted to publication

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