Verified lower eigenvalue bounds for self-adjoint differential operators

Xuefeng LIU

Research Institute for Science and Engineering
Waseda Univeristy, Japan

Cooperated with
Prof. S. Oishi, Waseda University, Japan
Prof. M. Plum, Karlsruhe Institute of Technology, Germany

SCAN'2014 Sep 25th, 2014

Various eigenvalue problems

- Laplace operator:

$$-\Delta u = \lambda u \text{ in } \Omega \quad \oplus \quad \text{boundary condition}$$

- Bi-harmonic operator:

$$\Delta^2 u = \lambda u \text{ or } \Delta^2 u = -\lambda \Delta u \text{ in } \Omega \quad \oplus \quad \text{boundary condition}$$

- Eigenvalue problems for Stokes's operator:

$$\begin{cases} -\Delta u + \nabla p = \lambda u \\ \text{div } u = 0 \end{cases} \text{ in } \Omega \quad \oplus \quad \text{boundary condition}$$

- Eigenvalue problem for Maxwell's operator:
  Find $E \in H_0(\textbf{rot}; \Omega)$ and $\lambda \in R$, s.t.,

$$(\textbf{rot}E, \textbf{rot}F) = \lambda(E, F) \quad \forall F \in H_0(\textbf{rot}; \Omega) \,.$$

Objective

Verified eigenvalue bounds for eigenvalue problems defined in the form:

$$\text{Find } u \in V \text{ and } \lambda > 0 \text{ s.t.} \quad M(u,v) = \lambda\, N(u,v), \quad \forall v \in V$$

where $M(\cdot,\cdot)$ and $N(\cdot,\cdot)$ are bilinear forms to be defined.

Objective

Verified eigenvalue bounds for eigenvalue problems defined in the form:

$$\text{Find } u \in V \text{ and } \lambda > 0 \text{ s.t.} \quad M(u,v) = \lambda\, N(u,v), \quad \forall v \in V$$

where $M(\cdot,\cdot)$ and $N(\cdot,\cdot)$ are bilinear forms to be defined.

- **Verified eigenvalue bounds** means mathematically correct lower and upper bounds. Thus interval arithmetic is adopted for this purpose.

Objective

<u>**Verified eigenvalue bounds**</u> for eigenvalue problems defined in the form:

$$\text{Find } u \in V \text{ and } \lambda > 0 \text{ s.t. } \quad M(u,v) = \lambda\, N(u,v), \quad \forall v \in V$$

where $M(\cdot,\cdot)$ and $N(\cdot,\cdot)$ are bilinear forms to be defined.

- **Verified eigenvalue bounds** means mathematically correct lower and upper bounds. Thus interval arithmetic is adopted for this purpose.
- Quantitative error estimation for various interpolation operators:

$$\|u - \Pi_h u\|_V \le C h^\alpha \|u\|_U \quad (C = ?)$$

Objective

Verified eigenvalue bounds for eigenvalue problems defined in the form:

$$\text{Find } u \in V \text{ and } \lambda > 0 \text{ s.t.} \quad M(u,v) = \lambda\, N(u,v), \quad \forall v \in V$$

where $M(\cdot,\cdot)$ and $N(\cdot,\cdot)$ are bilinear forms to be defined.

- **Verified eigenvalue bounds** means mathematically correct lower and upper bounds. Thus interval arithmetic is adopted for this purpose.
- Quantitative error estimation for various interpolation operators:

$$\|u - \Pi_h u\|_V \le C h^\alpha \|u\|_U \quad (C = ?)$$

- It is also needed in computer-assisted proof for non-linear equation solution verification, for example,

$$-\Delta u = u^2 \quad \text{in } \Omega$$

Objective

Verified eigenvalue bounds for eigenvalue problems defined in the form:

$$\text{Find } u \in V \text{ and } \lambda > 0 \text{ s.t.} \quad M(u,v) = \lambda\, N(u,v), \quad \forall v \in V$$

where $M(\cdot,\cdot)$ and $N(\cdot,\cdot)$ are bilinear forms to be defined.

- Such an abstract problem will include eigenvalue problems of the Laplace operator, the Bi-harmonic operator, the Stokes's operator and the Maxwell's operator.
- The famework has been successful in solving eigenvue problems of Δ, Δ^2, in 1D, 2D and 3D space.

Objective

Verified eigenvalue bounds for eigenvalue problems defined in the form:

$$\text{Find } u \in V \text{ and } \lambda > 0 \text{ s.t.} \quad M(u,v) = \lambda\, N(u,v), \quad \forall v \in V$$

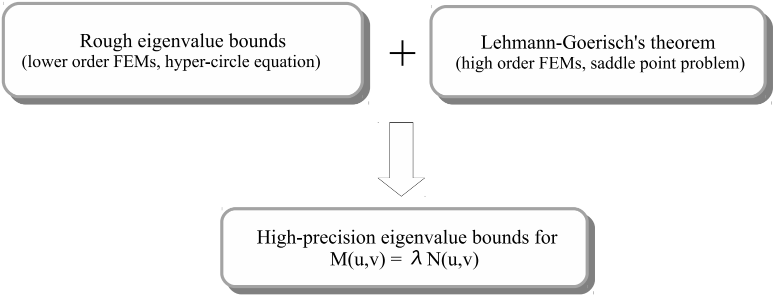
where $M(\cdot,\cdot)$ and $N(\cdot,\cdot)$ are bilinear forms to be defined.

- Such an abstract problem will include eigenvalue problems of the Laplace operator, the Bi-harmonic operator, the Stokes's operator and the Maxwell's operator.
- The famework has been successful in solving eigenvue problems of Δ, Δ^2, in 1D, 2D and 3D space.

Outline

1. Framework for high-precision eigenvalue bounds.
2. Rough eigenvalue bounds based on finite element method
3. High-precision eigenvalue bounds by applying Lehmann-Goerisch's theorem
4. Application and computation results

1. The framework for high-precision eigenvalue bounds



2. The main theorem to give rough lower and upper bounds

Preparation and assumptions

Preparation

- V: Hilbert space of real functions.
- V^h: Finite dimentional space; V^h may not be a subspace of V.

Assumption [Lehmann-Goerisch, 1960-1990]

A1 $M(u,v)$, $N(u,v)$ are symmetric bilinear forms over V and V^h; $M(u,u) \geq 0$, $N(u,u) \geq 0$; $N(u,u) = 0$ implies $u = 0$.

- Define $|\cdot|_M := \sqrt{M(\cdot,\cdot)}$, $|\cdot|_N := \sqrt{N(\cdot,\cdot)}$.

A2 There exist sequences $\{\phi_i\}_{i\in\mathbf{N}}$ and non-decreasing $\{\lambda_i\}_{i\in\mathbf{N}}$ such that $\phi_i \in V$, $\lambda_i \in \mathbf{R}$, $N(\phi_i,\phi_j) = \delta_{ij}$ for $i,j \in \mathbf{N}$,

$$M(f,\phi_i) = \lambda_i N(f,\phi_i) \qquad \text{for all } f \in V, i \in \mathbf{N}. \qquad (1)$$

$$N(f,f) = \sum_{i=1}^{\infty} (N(f,\phi_i))^2 \qquad \text{for all } f \in V. \qquad (2)$$

Upper eigenvalue bounds

Eigenvalue problem in V^h

Let $(\lambda_{h,k}, \phi_{h,k})_{k=1,\cdots,n}$ $(\lambda_{h,k} \leq \lambda_{h,k+1})$ be the eigen-pairs such that,

$$M(v_h, \phi_{h,k}) = \lambda_{h,k} N(v_h, \phi_{h,k}) \quad \forall v_h \in V^h .$$

Theorem (Upper eigenvalue bounds)

If $V^h \subset V$, then an upper bound for λ_k is given as,

$$\lambda_k \leq \lambda_{h,k} .$$

Upper eigenvalue bounds

Eigenvalue problem in V^h

Let $(\lambda_{h,k}, \phi_{h,k})_{k=1,\cdots,n}$ $(\lambda_{h,k} \le \lambda_{h,k+1})$ be the eigen-pairs such that,

$$M(v_h, \phi_{h,k}) = \lambda_{h,k} N(v_h, \phi_{h,k}) \quad \forall v_h \in V^h .$$

Theorem (Upper eigenvalue bounds)

If $V^h \subset V$, then an upper bound for λ_k is given as,

$$\lambda_k \le \lambda_{h,k} .$$

- A V^h satsifying $V^h \subset V$ is called conforming space .
- Usually, a conforming space is easy to construct.

Lower eigenvalue bounds

Theorem 1: Let $P_h : V \to V^h$ be a projection satisfying

$$M(u - P_h u, v_h) = 0, \quad \text{for all } v_h \in V^h$$

Moreover, suppose that an error estimation for P_h is given as,

$$|u - P_h u|_N \leq C_h |u - P_h u|_M \,.$$

Assertion: The lower bounds for eigenvalues are given as,

$$\lambda_{h,k}/(1 + \lambda_{h,k} C_h^2) \leq \lambda_k \quad (k = 1, 2, \cdots, n)\,.$$

Related results

Conforming case

- X. Liu and S. Oishi. Verified eigenvalue evaluation for the Laplacian over polygonal domains of arbitrary shape. SIAM J. Numer. Anal., 51(3): 1634-1654, 2013.

Non-conforming case

- Kobayashi: upper bound for interpolation constants (2010).
    - To consider the bound for the first eigenvalue of operators.
- Results of [Carstensen-Gallistl, 2013], [Carstensen-Gedicke, 2014]:
    - For the first eigenvalue: same result
    - For the rest eigenvalues: The same lower bounds are proposed but with a seperation condition:

$$C_h \leq (\sqrt{1+1/k} - 1)/\sqrt{\lambda_k}$$

Two tasks in the application of Theorem 1

1) Selection of proper space V^h and the projection P_h:

$$M(u - P_h u, v_h) = 0, \quad \text{for all } v_h \in V^h .$$

2) Explicit error estimation for P_h:

$$|u - P_h u|_N \leq C_h |u - P_h u|_M .$$

Two tasks in the application of Theorem 1

1) Selection of proper space V^h and the projection P_h:

$$M(u - P_h u, v_h) = 0, \quad \text{for all } v_h \in V^h .$$

2) Explicit error estimation for P_h:

$$|u - P_h u|_N \leq C_h |u - P_h u|_M .$$

- The finite element space will be adopted to deal with the eigenvalue problems of Laplace and Biharmonic operators.

Two tasks in the application of Theorem 1

1) Selection of proper space V^h and the projection P_h:

$$M(u - P_h u, v_h) = 0, \quad \text{for all } v_h \in V^h .$$

2) Explicit error estimation for P_h:

$$|u - P_h u|_N \le C_h |u - P_h u|_M .$$

- The finite element space will be adopted to deal with the eigenvalue problems of Laplace and Biharmonic operators.
- A locally defined interpolation operator Π_h which is also a projection operator will be a good candidate for P_h. That is, on each element K of triangulation $\mathcal{T}^h$,

$$(P_h u)|_K = \Pi_h (u|_K) .$$

Eigenvalue problem of Laplacian

Eigenvalue problem of Laplace operators

Eigenvalue problem for 2nd order differential operator

Assumption: Ω is a simply connected bounded domain.

$$-\Delta u = \lambda u, \quad u = 0 \text{ on } \partial\Omega,$$

Eigenvalue problem of Laplace operators

Eigenvalue problem for 2nd order differential operator

Assumption: Ω is a simply connected bounded domain.

$$-\Delta u = \lambda u, \quad u = 0 \text{ on } \partial\Omega,$$

Variational formulation:

Let $V := \{v \in H^1(\Omega) |\ u = 0 \text{ on } \partial\Omega\}$.

Find $u \in V$ and $\lambda \geq 0$ s.t. $$\int_\Omega \nabla u \cdot \nabla v dx = \lambda \int_\Omega uv dx \quad \forall v \in V .$$

Thus, we define: $M(u,v) = \int_\Omega \nabla u \cdot \nabla v dx, \quad N(u,v) = \int_\Omega uv dx$.

Eigenvalue problem of Laplace operators

Eigenvalue problem for 2nd order differential operator

Assumption: Ω is a simply connected bounded domain.

$$-\Delta u = \lambda u, \quad u = 0 \text{ on } \partial\Omega,$$

Variational formulation:

Let $V := \{v \in H^1(\Omega) | \; u = 0 \text{ on } \partial\Omega\}$.

$$\text{Find } u \in V \text{ and } \lambda \geq 0 \text{ s.t.} \quad \int_\Omega \nabla u \cdot \nabla v dx = \lambda \int_\Omega uv dx \quad \forall v \in V \, .$$

Thus, we define: $M(u,v) = \int_\Omega \nabla u \cdot \nabla v dx, \quad N(u,v) = \int_\Omega uv dx$.

- Classical methods have difficulty to bound eigenvalues for problems defined over a domain of general shape.

Eigenvalue bounds based on conforming FEMs

Eigenvalue problem in FEM spaces V^h

Lagrange FEM space: $V^h (\subset V)$

Let $\mathcal{T}^h$ be a triangulation of domain Ω. The function space V^h over $\mathcal{T}^h$ is consisted of function v_h such that,

1) $v_h|_K$ is linear function on each element $K \in \mathcal{T}^h$;
2) v_h is a continuous function over Ω.

The bilinear forms $M(\cdot,\cdot)$ and $N(\cdot,\cdot)$ over V^h:

$$M(u_h, v_h) = \int_\Omega \nabla u_h \cdot \nabla v_h dx, \quad N(u_h, v_h) = \int_\Omega u_h v_h dx \,.$$

Eigenvalue problem in FEM spaces V^h

Lagrange FEM space: $V^h(\subset V)$

Let $\mathcal{T}^h$ be a triangulation of domain Ω. The function space V^h over $\mathcal{T}^h$ is consisted of function v_h such that,

1) $v_h|_K$ is linear function on each element $K \in \mathcal{T}^h$;
2) v_h is a continuous function over Ω.

The bilinear forms $M(\cdot,\cdot)$ and $N(\cdot,\cdot)$ over V^h:

$$M(u_h, v_h) = \int_\Omega \nabla u_h \cdot \nabla v_h dx, \quad N(u_h, v_h) = \int_\Omega u_h v_h dx \, .$$

Eigenvalue problem in V^h: Find $u_h \in V^h$ and $\lambda_h \geq 0$ s.t.

$$M(u_h, v_h) = \lambda_h N(u_h, v_h) \quad \forall v_h \in V^h.$$

Error estimation for $P_h : V \to V^h$

Quantitative a priori error estimate on convex or non-convex domain:

Theorem [Liu-Oishi, SIMNM, 2013] Given $f \in L_2(\Omega)$, let $u \in H_0^1(\Omega)$ and $u_h \in V^h$ be the solutions of variational problems below, respectively,

$$(\nabla u, \nabla v) = (f, v) \quad \text{for } v \in H_0^1(\Omega)\,, \quad (\nabla u_h, \nabla v_h) = (f, v_h) \quad \text{for } v_h \in V^h(\Omega)\,.$$

Let $C_h := \sqrt{C_{0,h}^2 + \kappa_h^2}$, we have error estimates as below,

$$\|\nabla(u - u_h)\|_{L_2} \le C_h \|f\|_{L_2}, \quad \|u - u_h\|_{L_2} \le C_h^2 \|f\|_{L_2}$$

where $C_{0,h}$ has explicit value and κ_h is defined by

$$\kappa_h := \sup_{f_h \in X^h \setminus 0} \inf_{p_h \in W_{f_h}^h} \inf_{u_h \in S^h} \frac{\|p_h - \nabla u_h\|}{\|f_h\|}$$

Eigenvalue bounds based on non-conforming FEMs

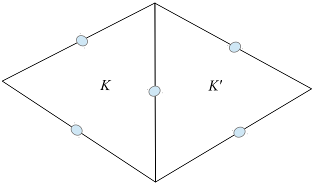
Eigenvalue problem in FEM spaces V^h

Crouzeix-Raviart FEM space: $V^h (\not\subset V)$

The function v_h of V^h satisfies,

1) v_h is linear on each element $K \in \mathcal{T}^h$;
2) $\int_e v_h ds$ is continuous on interior edges; $\int_e v_h ds = 0$ on boundary edges;

Function v_h is only continuous on the mid-points of interior edges.



Eigenvalue problem in FEM spaces V^h

Crouzeix-Raviart FEM space: $V^h(\not\subset V)$

The function v_h of V^h satisfies,

1) v_h is linear on each element $K \in \mathcal{T}^h$;
2) $\int_e v_h ds$ is continuous on interior edges; $\int_e v_h ds = 0$ on boundary edges;

Extend the bilinear forms M and N from V to $V + V^h$:

$$M(u_h, v_h) = \sum_{K \in \mathcal{T}^h} \int_K \nabla u_h \cdot \nabla v_h dx, \quad N(u_h, v_h) = \int_\Omega u_h v_h dx \, .$$

Eigenvalue problem in FEM spaces V^h

Crouzeix-Raviart FEM space: $V^h(\not\subset V)$

The function v_h of V^h satisfies,

1) v_h is linear on each element $K \in \mathcal{T}^h$;
2) $\int_e v_h ds$ is continuous on interior edges; $\int_e v_h ds = 0$ on boundary edges;

Extend the bilinear forms M and N from V to $V + V^h$:

$$M(u_h, v_h) = \sum_{K \in \mathcal{T}^h} \int_K \nabla u_h \cdot \nabla v_h dx, \quad N(u_h, v_h) = \int_\Omega u_h v_h dx \,.$$

Eigenvalue problem in V^h: Find $u_h \in V^h$ and $\lambda_h \geq 0$ s.t.

$$M(u_h, v_h) = \lambda_h N(u_h, v_h) \quad \forall v_h \in V^h.$$

Error estimation for projection P_h

Let h be the mesh size of triangulation for domain Ω.

$$\|u - P_h u\|_N \leq 0.19h \, \|u - P_h u\|_M \text{ for } u \in H^1(\Omega)$$

- The error constant $C_h = 0.19h$ is not depending on the maximum inner angle of triangle elements.

Related work: [Carsten-Gedicke, Mathematics of Computation, 2014] shows:

$$\|v - P_h v\|_N \leq 0.43955h\|v - P_h v\|_M$$

Lower eigenvalue bounds based on Theorem 1

Eigenvalue problem

$$-\Delta u = \lambda u, \quad u = 0 \text{ on } \partial\Omega,$$

Setting for application of Theorem 1

- $V = H_0^1(\Omega)$;
- V^h: Crouzeix-Rarviart FEM space ($V^h \not\subset V$);
- $M(u,v) := \sum_{K\in\mathcal{T}^h} \int_K \nabla u \cdot \nabla v dx$;
- $N(u,v) := \int_\Omega uv dx$;
- Projection $P_h := \Pi_h$:

$$M(u - P_h u, v_h) = 0 \text{ for } v_h \in V^h \,.$$

- Error estimation for P_h:

$$|u - P_h u|_N \le C_h |u - P_h u|_M \quad \left(C_h := \max_{K\in\mathcal{T}^h} C_e(K) \right)$$

Computation results for Laplacian

Example I: L-shaped domain

Domain: $\Omega : (0,2) \times (0,2) \setminus [1,2] \times [1,2]$.
Problem: Find $u \in H_0^1(\Omega)$ and $\lambda > 0$ such that,

$$-\Delta u = \lambda u \text{ in } \Omega, \quad u = 0 \text{ on } \Gamma \,.$$

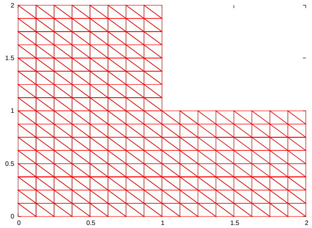


Figure : Triangulation of L-shape domain

Table : Conforming FEM (uniform mesh with $h = 1/32$)

λ_i	Lower bound	Exact	Upper bound	ReErr
1	9.5585	9.63972	9.6699	0.012
2	14.950	15.1973	15.225	0.018
3	19.326	19.7392	19.787	0.024
4	28.605	29.5215	29.626	0.035
5	30.866	31.9126	32.058	0.038

Table : Nonconforming FEM (uniform mesh with $h = 1/32$)

λ_i	Lower bound	Approx.	Exact.
1	9.6122	9.6155	9.63972
2	15.1833	15.1915	15.1973
3	19.7202	19.7339	19.7392
4	29.4697	29.5003	29.5215
5	31.7969	31.8326	31.9126

3. High-precision eigen-bounds from Lehmann-Goerisch's theorem

Challenges in desiring high-precision bounds

Challenges in desiring high-precision bounds

Take the eigenvalue problem of Δ as an example.

Kato's bound [Kato, 1949]

Let $\tilde{u} \in D(\Delta)$ be approximate eigenvector, and $\tilde{\lambda} := \|\nabla \tilde{u}\|^2 / \|\tilde{u}\|^2$ and $\sigma := \| -\Delta \tilde{u} - \tilde{\lambda}\tilde{u}\| / \|\tilde{u}\|$. Suppose that μ and ν satisfy, for certain n,

$$\lambda_{n-1} \le \mu < \tilde{\lambda} < \nu \le \lambda_{n+1}.$$

Thus,

$$\tilde{\lambda} - \frac{\sigma^2}{\nu - \tilde{\lambda}} \le \lambda_n \le \tilde{\lambda} + \frac{\sigma^2}{\tilde{\lambda} - \mu}$$

Challenges in desiring high-precision bounds

Take the eigenvalue problem of Δ as an example.

Kato's bound [Kato, 1949]

Let $\tilde{u} \in D(\Delta)$ be approximate eigenvector, and $\tilde{\lambda} := \|\nabla \tilde{u}\|^2 / \|\tilde{u}\|^2$ and $\sigma := \| - \Delta \tilde{u} - \tilde{\lambda}\tilde{u}\| / \|\tilde{u}\|$. Suppose that μ and ν satisfy, for certain n,

$$\lambda_{n-1} \le \mu < \tilde{\lambda} < \nu \le \lambda_{n+1}.$$

Thus,

$$\tilde{\lambda} - \frac{\sigma^2}{\nu - \tilde{\lambda}} \le \lambda_n \le \tilde{\lambda} + \frac{\sigma^2}{\tilde{\lambda} - \mu}$$

- A priori eigenvalue bounds μ and ν are needed;
- Well-constructed vector $\hat{u}$ can provide high-precision bounds;

○ Kato, T., On the Upper and Lower Bounds of Eigenvalues. Journal of the Physical Society of Japan, 4(4), 334-339. 1949

Remark

Various results related to eigenvalue bounds

- The original result of Kato can also deal with clustered eigenvalues, but it is not easy to use in practical computation since it requires the approximate eigenfunctions to be orthogonal to each other.

Remark

Various results related to eigenvalue bounds

- The original result of Kato can also deal with clustered eigenvalues, but it is not easy to use in practical computation since it requires the approximate eigenfunctions to be orthogonal to each other.
- Lehmann's theorem is almost the same as Kato's bound, but it can easily deal with clustered eigenvalues.

Remark

Various results related to eigenvalue bounds

- The original result of Kato can also deal with clustered eigenvalues, but it is not easy to use in practical computation since it requires the approximate eigenfunctions to be orthogonal to each other.
- Lehmann's theorem is almost the same as Kato's bound, but it can easily deal with clustered eigenvalues.
- Kato's bound or Lehmann's theorem requires that the approximate function $\hat{u}$ is smooth enough, while Lehmann-Goerisch's theorem relaxes such a condition.

Remark

Various results related to eigenvalue bounds

- The original result of Kato can also deal with clustered eigenvalues, but it is not easy to use in practical computation since it requires the approximate eigenfunctions to be orthogonal to each other.
- Lehmann's theorem is almost the same as Kato's bound, but it can easily deal with clustered eigenvalues.
- Kato's bound or Lehmann's theorem requires that the approximate function $\hat{u}$ is smooth enough, while Lehmann-Goerisch's theorem relaxes such a condition.

  For example, for the eigenvalue problem of $-\Delta$:
  - Kato's bound or Lehmann's theorem: $\hat{u} \in D(-\Delta)$.
  - Lehmann-Goerisch's theorem: $\hat{u} \in H^1(\Omega)$.

Computation examples for high-precision eigenvalue bounds

Examples of 2nd order operators

Eigenvalue problem of Laplacian

$$-\Delta u = \lambda u \quad \text{(boundary condition)}$$

Example: Poincare's constant [Liu-Oishi, JJIAM, 2013]

Average interpolation operator Π_0: Let K be a triangle,

$$\int_K \Pi_0 u - u ds = 0$$

Poincare's constant C_p:

$$\|u - \Pi_0 u\|_{L_2(K)} \le C_p |u|_{H^1(K)} \text{ for } u \in H^1(K)$$

Triangulation mesh

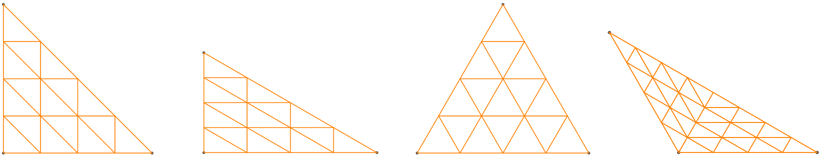
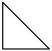
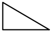
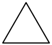
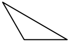


Figure : Triangulation of K (from left to right, $h = 0.25, 0.25, 0.25, 0.22$)

Example: Poincare's constant [Liu-Oishi, JJIAM, 2013]

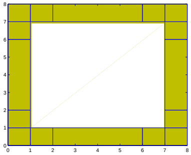
- Triangle with three vertices: $(0,0)$, $(1,0)$, (a,b)

Table : High-precision bound for constant C_p $(d=5)$

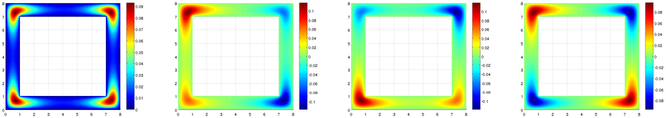
(a,b)	shape	h	λ_1	C_p
$(0,1)$		0.25	9.869604_{39}^{41}	0.318309886_{18}^{24}
$(0,\sqrt{3}/3)$		0.25	13.1594725_{23}^{36}	0.275664447_{70}^{83}
$(1/2,\sqrt{3}/2)$		0.25	17.5459633_{27}^{81}	0.238732414_{633}^{993}
$(-1/2,\sqrt{3}/2)$		0.22	7.1553_{26}^{53}	0.37383_{83}^{96}

Example: Non-convex domain [Liu-Okayama-Oishi, Comp. Math., 2014]

- Eigenvalue problem: $-\Delta u = \lambda u$ in Ω, $u = 0$ on $\partial\Omega$.
- Domain Ω: $(0,8)^2 \setminus [1,7]^2$;
- Rough a priori eigenvalue estimation: $\lambda_5 < 35.0 < \lambda_6$;
- Singular base function used around the re-entrant corners;
- Order of Lagrange FEM space L_h^d: $d = 10$.



λ_i	lower	upper
1	9.1602158	9.1602163
2	9.1700883	9.1700889
3	9.1700883	9.1700889
4	9.1805675	9.1805681



Eigenfunctions corresponding to the leading 4 eigenvalues

Examples of 4th order operators: Δ^2

Buckling plate eigenvalue problem

$$\Delta^2 u = -\lambda \Delta u, \quad u = \frac{\partial u}{\partial n} = 0 \text{ on } \partial\Omega,$$

Buckling plate eigenvalue problem

Unit square domain $\Omega := (0,1)^2$

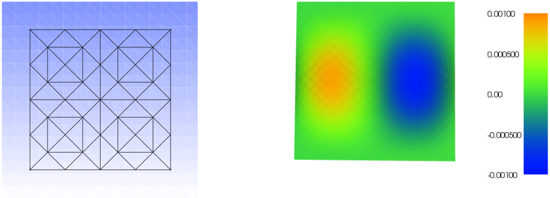


Figure : Left: triangulation for domain; Right: $\partial u / \partial x$

- Approximate eigenvalues
  $\lambda_1 \approx 52.3446989, \quad \lambda_2 \approx 92.1244138, \quad \lambda_3 = 92.1244138.$
- Eigenvalue bounds: (64 triangle elements; $d = 6$; $\rho = 85.0$)

$$52.34468 \leq \lambda_1 \leq 52.34470$$

Buckling plate eigenvalue problem

Unit triangle domain T: three vertices $(0,0)$, $(1,0)$, $(1,1)$.

Figure : Left: triangulation for domain; Right: $\partial u/\partial x$

- Approximate eigenvalues $\lambda_1 \approx 139.574, \quad \lambda_2 \approx 205.554, \quad \lambda_3 \approx 247.864.$
- Eigenvalue bounds: (32 triangle elements; $d = 6$; $\rho = 200.0$)

$$139.57361 \leq \lambda_1 \leq 139.57435$$

Summary

- We give a theorem to provide eigenvalue bounds for generally defined eigenvalue problems for self-adjoint operators:

$$\text{Find } u \in V \text{ and } \lambda \in R, \quad M(u,v) = \lambda N(u,v) \quad \forall v \in V.$$

Summary

- We give a theorem to provide eigenvalue bounds for generally defined eigenvalue problems for self-adjoint operators:

$$\text{Find } u \in V \text{ and } \lambda \in R, \quad M(u,v) = \lambda N(u,v) \quad \forall v \in V.$$

- High-precision eigenvalue bounds can be obtained as follows.