Verified lower eigenvalue bounds for self-adjoint differential operators

Xuefeng LIU

Research Institute for Science and Engineering
Waseda University, Japan

Cooperated with
Prof. S. Oishi, Waseda University, Japan
Prof. M. Plum, Karlsruhe Institute of Technology, Germany

SCAN’2014 Sep 25th, 2014
Various eigenvalue problems

- Laplace operator:
  \[-\Delta u = \lambda u \text{ in } \Omega \quad \oplus \quad \text{boundary condition}\]

- Bi-harmonic operator:
  \[\Delta^2 u = \lambda u \text{ or } \Delta^2 u = -\lambda \Delta u \text{ in } \Omega \quad \oplus \quad \text{boundary condition}\]

- Eigenvalue problems for Stokes’s operator:
  \[
  \begin{cases}
  -\Delta u + \nabla p = \lambda u \\
  \text{div } u = 0
  \end{cases}
  \text{ in } \Omega \quad \oplus \quad \text{boundary condition}
  \]

- Eigenvalue problem for Maxwell’s operator:
  Find \(E \in H_0(\operatorname{rot}; \Omega)\) and \(\lambda \in \mathbb{R}\), s.t.,
  \[(\operatorname{rot} E, \operatorname{rot} F) = \lambda (E, F) \quad \forall F \in H_0(\operatorname{rot}; \Omega)\].
Objective

**Verified eigenvalue bounds** for eigenvalue problems defined in the form:

Find $u \in V$ and $\lambda > 0$ s.t. $M(u, v) = \lambda N(u, v)$, $\forall v \in V$

where $M(\cdot, \cdot)$ and $N(\cdot, \cdot)$ are bilinear forms to be defined.
Objective

**Verified eigenvalue bounds** for eigenvalue problems defined in the form:

$$\text{Find } u \in V \text{ and } \lambda > 0 \text{ s.t. } M(u, v) = \lambda N(u, v), \quad \forall v \in V$$

where $M(\cdot, \cdot)$ and $N(\cdot, \cdot)$ are bilinear forms to be defined.

- **Verified eigenvalue bounds** means mathematically correct lower and upper bounds. Thus interval arithmetic is adopted for this purpose.
**Objective**

**Verified eigenvalue bounds** for eigenvalue problems defined in the form:

\[
\text{Find } u \in V \text{ and } \lambda > 0 \text{ s.t. } M(u, v) = \lambda N(u, v), \quad \forall v \in V
\]

where \(M(\cdot, \cdot)\) and \(N(\cdot, \cdot)\) are bilinear forms to be defined.

- **Verified eigenvalue bounds** means mathematically correct lower and upper bounds. Thus interval arithmetic is adopted for this purpose.
- Quantitative error estimation for various interpolation operators:

  \[
  \|u - \Pi_h u\|_V \leq C h^\alpha \|u\|_U \quad (C = ?)
  \]
Objective

**Verified eigenvalue bounds** for eigenvalue problems defined in the form:

Find $u \in V$ and $\lambda > 0$ s.t. $M(u, v) = \lambda N(u, v), \forall v \in V$

where $M(\cdot, \cdot)$ and $N(\cdot, \cdot)$ are bilinear forms to be defined.

- **Verified eigenvalue bounds** means mathematically correct lower and upper bounds. Thus interval arithmetic is adopted for this purpose.
- Quantitative error estimation for various interpolation operators:
  \[
  \|u - \Pi_h u\|_V \leq Ch^\alpha \|u\|_U \quad (C = ?)
  \]
- It is also needed in computer-assisted proof for non-linear equation solution verification, for example,
  \[-\Delta u = u^2 \quad \text{in } \Omega\]
Objective

**Verified eigenvalue bounds** for eigenvalue problems defined in the form:

\[
\text{Find } u \in V \text{ and } \lambda > 0 \text{ s.t. } M(u, v) = \lambda N(u, v), \quad \forall v \in V
\]

where \( M(\cdot, \cdot) \) and \( N(\cdot, \cdot) \) are bilinear forms to be defined.

- Such an abstract problem will include eigenvalue problems of the Laplace operator, the Bi-harmonic operator, the Stokes’s operator and the Maxwell’s operator.
- The framework has been successful in solving eigenvue problems of \( \Delta, \Delta^2 \), in 1D, 2D and 3D space.
Objective

**Verified eigenvalue bounds** for eigenvalue problems defined in the form:

\[
\text{Find } u \in V \text{ and } \lambda > 0 \text{ s.t. } M(u, v) = \lambda N(u, v), \quad \forall v \in V
\]

where \( M(\cdot, \cdot) \) and \( N(\cdot, \cdot) \) are bilinear forms to be defined.

- Such an abstract problem will include eigenvalue problems of the Laplace operator, the Bi-harmonic operator, the Stokes’s operator and the Maxwell’s operator.
- The framework has been successful in solving eigenvue problems of \( \Delta, \Delta^2 \), in 1D, 2D and 3D space.
Outline

2. Rough eigenvalue bounds based on finite element method
3. High-precision eigenvalue bounds by applying Lehmann-Goerisch’s theorem
4. Application and computation results
1. The framework for high-precision eigenvalue bounds

- Rough eigenvalue bounds (lower order FEMs, hyper-circle equation)
- Lehmann-Goerisch's theorem (high order FEMs, saddle point problem)

High-precision eigenvalue bounds for
\[ M(u,v) = \lambda N(u,v) \]
2. The main theorem to give rough lower and upper bounds
Preparation and assumptions

Preparation

- \( V \): Hilbert space of real functions.
- \( V^h \): Finite dimensional space; \( V^h \) may not be a subspace of \( V \).

Assumption [Lehmann-Goerisch, 1960-1990]

A1 \( M(u, v), N(u, v) \) are symmetric bilinear forms over \( V \) and \( V^h \); \( M(u, u) \geq 0, N(u, u) \geq 0 \); \( N(u, u) = 0 \) implies \( u = 0 \).

- Define \( | \cdot |_M := \sqrt{M(\cdot, \cdot)} , | \cdot |_N := \sqrt{N(\cdot, \cdot)} \).

A2 There exist sequences \( \{ \phi_i \}_{i \in \mathbb{N}} \) and non-decreasing \( \{ \lambda_i \}_{i \in \mathbb{N}} \) such that \( \phi_i \in V, \lambda_i \in \mathbb{R}, N(\phi_i, \phi_j) = \delta_{ij} \) for \( i, j \in \mathbb{N} \),

\[
M(f, \phi_i) = \lambda_i N(f, \phi_i) \quad \text{for all } f \in V, i \in \mathbb{N}. \tag{1}
\]

\[
N(f, f) = \sum_{i=1}^{\infty} (N(f, \phi_i))^2 \quad \text{for all } f \in V. \tag{2}
\]
Upper eigenvalue bounds

Eigenvalue problem in $V^h$

Let $(\lambda_{h,k}, \phi_{h,k})_{k=1, \ldots, n}$ $(\lambda_{h,k} \leq \lambda_{h,k+1})$ be the eigen-pairs such that,

$$M(v_h, \phi_{h,k}) = \lambda_{h,k} N(v_h, \phi_{h,k}) \quad \forall v_h \in V^h.$$ 

Theorem (Upper eigenvalue bounds)

*If $V^h \subset V$, then an upper bound for $\lambda_k$ is given as,

$$\lambda_k \leq \lambda_{h,k}.$$*
Upper eigenvalue bounds

Eigenvalue problem in $V^h$

Let $(\lambda_{h,k}, \phi_{h,k})_{k=1,\ldots,n}$ $(\lambda_{h,k} \leq \lambda_{h,k+1})$ be the eigen-pairs such that,

$$M(v_h, \phi_{h,k}) = \lambda_{h,k} N(v_h, \phi_{h,k}) \quad \forall v_h \in V^h.$$ 

Theorem (Upper eigenvalue bounds)

If $V^h \subset V$, then an upper bound for $\lambda_k$ is given as,

$$\lambda_k \leq \lambda_{h,k}.$$ 

- A $V^h$ satisfying $V^h \subset V$ is called **conforming space**.

- Usually, a conforming space is easy to construct.
Lower eigenvalue bounds

**Theorem 1:** Let $P_h : V \rightarrow V^h$ be a projection satisfying

$$M(u - P_h u, v_h) = 0, \quad \text{for all } v_h \in V^h$$

Moreover, suppose that an error estimation for $P_h$ is given as,

$$|u - P_h u|_N \leq C_h |u - P_h u|_M.$$

**Assertion:** The lower bounds for eigenvalues are given as,

$$\lambda_{h,k}/(1 + \lambda_{h,k} C_h^2) \leq \lambda_k \quad (k = 1, 2, \cdots, n).$$
Related results

Conforming case


Non-conforming case

  - To consider the bound for the first eigenvalue of operators.

- Results of [Carstensen-Gallistl, 2013], [Carstensen-Gedicke, 2014]:
  - For the first eigenvalue: same result
  - For the rest eigenvalues: The same lower bounds are proposed but with a separation condition:

$$C_h \leq \left( \sqrt{1 + 1/k} - 1 \right) / \sqrt{\lambda_k}$$
Two tasks in the application of Theorem 1

1) Selection of proper space $V^h$ and the projection $P_h$:

$$M(u - P_h u, v_h) = 0, \quad \text{for all } v_h \in V^h.$$ $

2) Explicit error estimation for $P_h$:

$$|u - P_h u|_N \leq C_h |u - P_h u|_M.$$
Two tasks in the application of Theorem 1

1) Selection of proper space $V^h$ and the projection $P_h$:

$$M(u - P_h u, v_h) = 0, \quad \text{for all } v_h \in V^h.$$  

2) Explicit error estimation for $P_h$:

$$|u - P_h u|_N \leq C_h |u - P_h u|_M.$$  

The finite element space will be adopted to deal with the eigenvalue problems of Laplace and Biharmonic operators.
Two tasks in the application of Theorem 1

1) Selection of proper space $V^h$ and the projection $P_h$:

$$M(u - P_h u, v_h) = 0, \quad \text{for all } v_h \in V^h.$$ 

2) Explicit error estimation for $P_h$:

$$|u - P_h u|_N \leq C_h |u - P_h u|_M.$$ 

- The finite element space will be adopted to deal with the eigenvalue problems of Laplace and Biharmonic operators.
- A locally defined interpolation operator $\Pi_h$ which is also a projection operator will be a good candidate for $P_h$. That is, on each element $K$ of triangulation $T^h$,

$$(P_h u)|_K = \Pi_h (u|_K).$$
Eigenvalue problem of Laplacian
Eigenvalue problem of Laplace operators

Eigenvalue problem for 2nd order differential operator

Assumption: $\Omega$ is a simply connected bounded domain.

$$-\Delta u = \lambda u, \quad u = 0 \text{ on } \partial \Omega,$$
Eigenvalue problem of Laplace operators

Eigenvalue problem for 2nd order differential operator

Assumption: \( \Omega \) is a simply connected bounded domain.

\[-\Delta u = \lambda u, \quad u = 0 \text{ on } \partial \Omega,\]

Variational formulation:

Let \( V := \{ v \in H^1(\Omega) | \ u = 0 \text{ on } \partial \Omega \} \).

Find \( u \in V \) and \( \lambda \geq 0 \) s.t.

\[ \int_\Omega \nabla u \cdot \nabla v dx = \lambda \int_\Omega uv dx \quad \forall v \in V. \]

Thus, we define:

\[ M(u, v) = \int_\Omega \nabla u \cdot \nabla v dx, \quad N(u, v) = \int_\Omega uv dx. \]
Eigenvalue problem of Laplace operators

Eigenvalue problem for 2nd order differential operator
Assumption: \( \Omega \) is a simply connected bounded domain.

\[-\Delta u = \lambda u, \quad u = 0 \text{ on } \partial \Omega,\]

Variational formulation:
Let \( V := \{ v \in H^1(\Omega) | u = 0 \text{ on } \partial \Omega \} \).

Find \( u \in V \) and \( \lambda \geq 0 \) s.t.

\[\int_{\Omega} \nabla u \cdot \nabla v dx = \lambda \int_{\Omega} uv dx \quad \forall v \in V.\]

Thus, we define:

\[M(u, v) = \int_{\Omega} \nabla u \cdot \nabla v dx, \quad N(u, v) = \int_{\Omega} uv dx.\]

Classical methods have difficulty to bound eigenvalues for problems defined over a domain of general shape.
Eigenvalue bounds based on conforming FEMs
Eigenvalue problem in FEM spaces $V^h$

Lagrange FEM space: $V^h(\subset V)$

Let $T^h$ be a triangulation of domain $\Omega$. The function space $V^h$ over $T^h$ is consisted of function $v_h$ such that,

1) $v_h|_K$ is linear function on each element $K \in T^h$;
2) $v_h$ is a continuous function over $\Omega$.

The bilinear forms $M(\cdot, \cdot)$ and $N(\cdot, \cdot)$ over $V^h$:

$$M(u_h, v_h) = \int_\Omega \nabla u_h \cdot \nabla v_h \, dx, \quad N(u_h, v_h) = \int_\Omega u_h v_h \, dx.$$
Eigenvalue problem in FEM spaces $V^h$

**Lagrange FEM space:** $V^h(\subset V)$

Let $\mathcal{T}^h$ be a triangulation of domain $\Omega$. The function space $V^h$ over $\mathcal{T}^h$ is consisted of function $v_h$ such that,

1) $v_h|_K$ is linear function on each element $K \in \mathcal{T}^h$;
2) $v_h$ is a continuous function over $\Omega$.

The bilinear forms $M(\cdot, \cdot)$ and $N(\cdot, \cdot)$ over $V^h$: 

$$M(u_h, v_h) = \int_{\Omega} \nabla u_h \cdot \nabla v_h \, dx, \quad N(u_h, v_h) = \int_{\Omega} u_h v_h \, dx.$$ 

**Eigenvalue problem in $V^h$:** Find $u_h \in V^h$ and $\lambda_h \geq 0$ s.t.

$$M(u_h, v_h) = \lambda_h N(u_h, v_h) \quad \forall v_h \in V^h.$$
Error estimation for $P_h : V \to V^h$

Quantitative a priori error estimate on convex or non-convex domain:

**Theorem** [Liu-Oishi, SIMNM, 2013] Given $f \in L_2(\Omega)$, let $u \in H^1_0(\Omega)$ and $u_h \in V^h$ be the solutions of variational problems below, respectively,

$$(\nabla u, \nabla v) = (f, v) \quad \text{for } v \in H^1_0(\Omega), \quad (\nabla u_h, \nabla v_h) = (f, v_h) \quad \text{for } v_h \in V^h(\Omega).$$

Let $C_h := \sqrt{C_{0,h}^2 + \kappa_h^2}$, we have error estimates as below,

$$\|\nabla (u - u_h)\|_{L^2} \leq C_h \|f\|_{L^2}, \quad \|u - u_h\|_{L^2} \leq C_h^2 \|f\|_{L^2}$$

where $C_{0,h}$ has explicit value and $\kappa_h$ is defined by

$$\kappa_h := \sup_{f_h \in X^h \setminus 0} \inf_{p_h \in W_{f_h}^h} \inf_{u_h \in S^h} \frac{\|p_h - \nabla u_h\|}{\|f_h\|}$$
Eigenvalue bounds based on non-conforming FEMs
Eigenvalue problem in FEM spaces $V^h$

Crouzeix-Raviart FEM space: $V^h(\not\subset V)$

The function $v_h$ of $V^h$ satisfies,

1) $v_h$ is linear on each element $K \in \mathcal{T}^h$;
2) $\int_e v_h ds$ is continuous on interior edges; $\int_e v_h ds = 0$ on boundary edges;

Function $v_h$ is only continuous on the mid-points of interior edges.
Eigenvalue problem in FEM spaces $V^h$

Crouzeix-Raviart FEM space: $V^h(\not\in V)$

The function $v_h$ of $V^h$ satisfies,

1) $v_h$ is linear on each element $K \in T^h$;
2) $\int_{e} v_h ds$ is continuous on interior edges; $\int_{e} v_h ds = 0$ on boundary edges;

Extend the bilinear forms $M$ and $N$ from $V$ to $V + V^h$:

$$M(u_h, v_h) = \sum_{K \in T^h} \int_{K} \nabla u_h \cdot \nabla v_h dx, \quad N(u_h, v_h) = \int_{\Omega} u_h v_h dx.$$
Eigenvalue problem in FEM spaces $V^h$

Crouzeix-Raviart FEM space: $V^h(\not \subset V)$

The function $v_h$ of $V^h$ satisfies,

1) $v_h$ is linear on each element $K \in \mathcal{T}^h$;
2) $\int_e v_h ds$ is continuous on interior edges; $\int_e v_h ds = 0$ on boundary edges;

Extend the bilinear forms $M$ and $N$ from $V$ to $V + V^h$:

$$M(u_h, v_h) = \sum_{K \in \mathcal{T}^h} \int_K \nabla u_h \cdot \nabla v_h dx, \quad N(u_h, v_h) = \int_\Omega u_h v_h dx.$$  

Eigenvalue problem in $V^h$: Find $u_h \in V^h$ and $\lambda_h \geq 0$ s.t.

$$M(u_h, v_h) = \lambda_h N(u_h, v_h) \quad \forall v_h \in V^h.$$
Error estimation for projection $P_h$

Let $h$ be the mesh size of triangulation for domain $\Omega$.

$$\|u - P_h u\|_N \leq 0.19h \|u - P_h u\|_M$$

for $u \in H^1(\Omega)$

- The error constant $C_h = 0.19h$ is not depending on the maximum inner angle of triangle elements.

**Related work:** [Carsten-Gedicke, Mathematics of Computation, 2014] shows:

$$\|v - P_h v\|_N \leq 0.43955h\|v - P_h v\|_M$$
Lower eigenvalue bounds based on Theorem 1

Eigenvalue problem

\[-\Delta u = \lambda u, \quad u = 0 \text{ on } \partial \Omega,\]

Setting for application of Theorem 1

- \( V = H^1_0(\Omega); \)
- \( V^h: \) Crouzeix-Rarviart FEM space \((V^h \not\subset V);\)
- \( M(u, v) := \sum_{K \in T^h} \int_K \nabla u \cdot \nabla v dx; \)
- \( N(u, v) := \int_\Omega uv dx; \)
- Projection \( P_h := \Pi_h: \)
  \[
  M(u - P_h u, v_h) = 0 \text{ for } v_h \in V^h. 
  \]
- Error estimation for \( P_h: \)

\[ |u - P_h u|_N \leq C_h |u - P_h u|_M \quad \left( C_h := \max_{K \in T^h} C_e(K) \right) \]
Computation results for Laplacian
Example I: L-shaped domain

Domain: $\Omega : (0, 2) \times (0, 2) \setminus [1, 2] \times [1, 2]$.

Problem: Find $u \in H^1_0(\Omega)$ and $\lambda > 0$ such that,

$$-\Delta u = \lambda u \quad \text{in} \quad \Omega, \quad u = 0 \quad \text{on} \quad \Gamma.$$ 

\[ \text{Figure: Triangulation of L-shape domain} \]
### Table: Conforming FEM (uniform mesh with $h = 1/32$)

<table>
<thead>
<tr>
<th>$\lambda_i$</th>
<th>Lower bound</th>
<th>Exact</th>
<th>Upper bound</th>
<th>ReErr</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>9.5585</td>
<td>9.63972</td>
<td>9.6699</td>
<td>0.012</td>
</tr>
<tr>
<td>2</td>
<td>14.950</td>
<td>15.1973</td>
<td>15.225</td>
<td>0.018</td>
</tr>
<tr>
<td>3</td>
<td>19.326</td>
<td>19.7392</td>
<td>19.787</td>
<td>0.024</td>
</tr>
<tr>
<td>4</td>
<td>28.605</td>
<td>29.5215</td>
<td>29.626</td>
<td>0.035</td>
</tr>
<tr>
<td>5</td>
<td>30.866</td>
<td>31.9126</td>
<td>32.058</td>
<td>0.038</td>
</tr>
</tbody>
</table>

### Table: Nonconforming FEM (uniform mesh with $h = 1/32$)

<table>
<thead>
<tr>
<th>$\lambda_i$</th>
<th>Lower bound</th>
<th>Approx.</th>
<th>Exact.</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>9.6122</td>
<td>9.6155</td>
<td>9.63972</td>
</tr>
<tr>
<td>2</td>
<td>15.1833</td>
<td>15.1915</td>
<td>15.1973</td>
</tr>
<tr>
<td>3</td>
<td>19.7202</td>
<td>19.7339</td>
<td>19.7392</td>
</tr>
<tr>
<td>4</td>
<td>29.4697</td>
<td>29.5003</td>
<td>29.5215</td>
</tr>
<tr>
<td>5</td>
<td>31.7969</td>
<td>31.8326</td>
<td>31.9126</td>
</tr>
</tbody>
</table>
3. High-precision eigen-bounds from Lehmann-Goerisch’s theorem
Challenges in desiring high-precision bounds
Challenges in desiring high-precision bounds

Take the eigenvalue problem of $\Delta$ as an example.

**Kato’s bound [Kato, 1949]**

Let $\tilde{u} \in D(\Delta)$ be approximate eigenvector, and $\tilde{\lambda} := \|\nabla \tilde{u}\|^2 / \|\tilde{u}\|^2$ and $\sigma := \| - \Delta \tilde{u} - \tilde{\lambda} \tilde{u} \| / \|\tilde{u}\|$. Suppose that $\mu$ and $\nu$ satisfy, for certain $n$,

$$\lambda_{n-1} \leq \mu < \tilde{\lambda} < \nu \leq \lambda_{n+1}. $$

Thus,

$$\tilde{\lambda} - \frac{\sigma^2}{\nu - \tilde{\lambda}} \leq \lambda_n \leq \tilde{\lambda} + \frac{\sigma^2}{\tilde{\lambda} - \mu}. $$
Challenges in desiring high-precision bounds

Take the eigenvalue problem of $\Delta$ as an example.

**Kato’s bound [Kato, 1949]**

Let $\tilde{u} \in D(\Delta)$ be approximate eigenvector, and $\tilde{\lambda} := \|\nabla \tilde{u}\|^2 / \|\tilde{u}\|^2$ and $\sigma := \| - \Delta \tilde{u} - \tilde{\lambda} \tilde{u} / \|\tilde{u}\|$. Suppose that $\mu$ and $\nu$ satisfy, for certain $n$,

$$\lambda_{n-1} \leq \mu < \tilde{\lambda} < \nu \leq \lambda_{n+1}.$$  

Thus,

$$\tilde{\lambda} - \frac{\sigma^2}{\nu - \tilde{\lambda}} \leq \lambda_n \leq \tilde{\lambda} + \frac{\sigma^2}{\tilde{\lambda} - \mu}$$

- A priori eigenvalue bounds $\mu$ and $\nu$ are needed;
- Well-constructed vector $\hat{u}$ can provide high-precision bounds;

Remark

Various results related to eigenvalue bounds

- The original result of Kato can also deal with clustered eigenvalues, but it is not easy to use in practical computation since it requires the approximate eigenfunctions to be orthogonal to each other.
Remark

Various results related to eigenvalue bounds

- The original result of Kato can also deal with clustered eigenvalues, but it is not easy to use in practical computation since it requires the approximate eigenfunctions to be orthogonal to each other.
- Lehmann’s theorem is almost the same as Kato’s bound, but it can easily deal with clustered eigenvalues.
Remark

Various results related to eigenvalue bounds

- The original result of Kato can also deal with clustered eigenvalues, but it is not easy to use in practical computation since it requires the approximate eigenfunctions to be orthogonal to each other.
- Lehmann’s theorem is almost the same as Kato’s bound, but it can easily deal with clustered eigenvalues.
- Kato’s bound or Lehmann’s theorem requires that the approximate function $\hat{u}$ is smooth enough, while Lehmann-Goerisch’s theorem relaxes such a condition.
Remark

Various results related to eigenvalue bounds

- The original result of Kato can also deal with clustered eigenvalues, but it is not easy to use in practical computation since it requires the approximate eigenfunctions to be orthogonal to each other.
- Lehmann’s theorem is almost the same as Kato’s bound, but it can easily deal with clustered eigenvalues.
- Kato’s bound or Lehmann’s theorem requires that the approximate function $\hat{u}$ is smooth enough, while Lehmann-Goerisch’s theorem relaxes such a condition.

For example, for the eigenvalue problem of $-\Delta$:

- Kato's bound or Lehmann's theorem: $\hat{u} \in D(-\Delta)$.
- Lehmann-Goerisch's theorem: $\hat{u} \in H^1(\Omega)$. 
Computation examples for high-precision eigenvalue bounds
Examples of 2nd order operators

Eigenvalue problem of Laplacian

\[ -\Delta u = \lambda u \quad \text{(boundary condition)} \]
Example: Poincare’s constant [Liu-Oishi, JJIAM, 2013]

**Average interpolation operator** $\Pi_0$: Let $K$ be a triangle,

$$\int_K \Pi_0 u - u ds = 0$$

**Poincare’s constant** $C_p$:

$$\|u - \Pi_0 u\|_{L^2(K)} \leq C_p |u|_{H^1(K)} \text{ for } u \in H^1(K)$$

**Triangulation mesh**

![Triangulation mesh images]

**Figure:** Triangulation of $K$ (from left to right, $h = 0.25, 0.25, 0.25, 0.22$)
Example: Poincare’s constant [Liu-Oishi, JJIAM, 2013]

- Triangle with three vertices: $(0, 0), (1, 0), (a, b)$

<table>
<thead>
<tr>
<th>$(a, b)$</th>
<th>shape</th>
<th>$h$</th>
<th>$\lambda_1$</th>
<th>$C_p$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$(0, 1)$</td>
<td>[\triangle]</td>
<td>0.25</td>
<td>$9.869604,\frac{41}{39}$</td>
<td>$0.318309886,\frac{24}{18}$</td>
</tr>
<tr>
<td>$(0, \sqrt{3}/3)$</td>
<td>[\triangle]</td>
<td>0.25</td>
<td>$13.1594725,\frac{36}{23}$</td>
<td>$0.275664447,\frac{83}{70}$</td>
</tr>
<tr>
<td>$(1/2, \sqrt{3}/2)$</td>
<td>[\triangle]</td>
<td>0.25</td>
<td>$17.5459633,\frac{81}{27}$</td>
<td>$0.238732414,\frac{993}{633}$</td>
</tr>
<tr>
<td>$(-1/2, \sqrt{3}/2)$</td>
<td>[\triangle]</td>
<td>0.22</td>
<td>$7.1553,\frac{53}{26}$</td>
<td>$0.37383,\frac{96}{83}$</td>
</tr>
</tbody>
</table>
Example: Non-convex domain [Liu-Okayama-Oishi, Comp. Math., 2014]

- Eigenvalue problem: \(-\Delta u = \lambda u\) in \(\Omega\), \(u = 0\) on \(\partial\Omega\).
- Domain \(\Omega\): \((0, 8)^2 \setminus [1, 7]^2\);
- Rough a priori eigenvalue estimation: \(\lambda_5 < 35.0 < \lambda_6\);
- Singular base function used around the re-entrant corners;
- Order of Lagrange FEM space \(L^d_h\): \(d = 10\).

<table>
<thead>
<tr>
<th>(\lambda_i)</th>
<th>lower</th>
<th>upper</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>9.1602158</td>
<td>9.1602163</td>
</tr>
<tr>
<td>2</td>
<td>9.1700883</td>
<td>9.1700889</td>
</tr>
<tr>
<td>3</td>
<td>9.1700883</td>
<td>9.1700889</td>
</tr>
<tr>
<td>4</td>
<td>9.1805675</td>
<td>9.1805681</td>
</tr>
</tbody>
</table>

Eigenfunctions corresponding to the leading 4 eigenvalues
Examples of 4th order operators: $\Delta^2$

**Buckling plate eigenvalue problem**

$$\Delta^2 u = -\lambda \Delta u, \quad u = \frac{\partial u}{\partial n} = 0 \text{ on } \partial \Omega,$$
Buckling plate eigenvalue problem

Unit square domain \( \Omega := (0, 1)^2 \)

**Figure:** Left: triangulation for domain; Right: \( \partial u / \partial x \)

- Approximate eigenvalues
  \( \lambda_1 \approx 52.3446989, \quad \lambda_2 \approx 92.1244138, \quad \lambda_3 = 92.1244138. \)

- Eigenvalue bounds: \( (64 \text{ triangle elements}; \; d = 6; \; \rho = 85.0) \)

\[
52.34468 \leq \lambda_1 \leq 52.34470
\]
Buckling plate eigenvalue problem

Unit triangle domain $T$: three vertices $(0,0)$, $(1,0)$, $(1,1)$.

Figure: Left: triangulation for domain; Right: $\partial u/\partial x$

- Approximate eigenvalues $\lambda_1 \approx 139.574$, $\lambda_2 \approx 205.554$, $\lambda_3 \approx 247.864$.
- Eigenvalue bounds: (32 triangle elements; $d = 6$; $\rho = 200.0$)

$$139.57361 \leq \lambda_1 \leq 139.57435$$
Summary

- We give a theorem to provide eigenvalue bounds for generally defined eigenvalue problems for self-adjoint operators:

  \[ \text{Find } u \in V \text{ and } \lambda \in \mathbb{R}, \quad M(u, v) = \lambda N(u, v) \quad \forall v \in V. \]
Summary

- We give a theorem to provide eigenvalue bounds for generally defined eigenvalue problems for self-adjoint operators:
  \[ \text{Find } u \in V \text{ and } \lambda \in \mathbb{R}, \quad M(u, v) = \lambda N(u, v) \quad \forall v \in V. \]

- High-precision eigenvalue bounds can be obtained as follows.

```
Rough eigenvalue bounds
(lower order FEMs, hyper-circle equation)

+ Lehmann-Goerisch's theorem
(high order FEMs, saddle point problem)

High-precision eigenvalue bounds for
M(u,v) = \lambda N(u,v)
```