The Forthcoming IEEE1788 Standard for Interval Arithmetic

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1. What intervals are and do
2. Why intervals need new algorithms
3. The need for a standard
4. 1788 Interval Principles
   - Definition of an interval
   - Levels
   - Inter-level maps
5. Exception handling
6. Difficulties
7. Current state
Outline

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Interval Arithmetic (IA) implements “validated” (=“verified”) numerics—it can enclose solution components \( x \) of a problem in an interval, i.e. between lower and upper bounds \( x \in x = [\underline{x}, \overline{x}] = \{ t \in \mathbb{R} | \underline{x} \leq t \leq \overline{x} \} \), even in finite-precision arithmetic.

E.g. it makes Brouwer’s fixed point theorem:

If \( K \subset \mathbb{R}^n \) is compact convex, and function \( f \) is everywhere defined & continuous on \( K \), and \( f(K) \subseteq K \), then \( f \) has a fixpoint in \( K \)

constructive in the sense that sufficient conditions for “everywhere defined & continuous” can be found while computing \( f \).

IA’s history: back to Archimedes (?) but mostly 20th century:
Sunaga (Japan), Rall (USA), et al.
Modern theory R. Moore (1966), e.g. validated ODE solver.

Current significant validated software exists for: global optimisation; large sparse linear systems; particle beam design for LHC, . . .
The basic idea

- Interval operations take all combinations of points in the inputs, i.e.
  \[ x \circ y = \{ x \circ y \mid x \in x \text{ and } y \in y \}, \]  
  where \( \circ \) is one of \( \{+ - \times \div\} \).

  For \( \div \) don’t allow \( 0 \in y \) for now. In finite precision round outward.

- Fundamental Theorem of Interval Arithmetic
  If function \( f(x_1, \ldots, x_n) \), defined by an expression, is evaluated with
  interval operations on interval inputs to get \( y = f(x_1, \ldots, x_n) \) then
  \[ y \supseteq \text{range of } f \text{ over box } x_1 \times \cdots \times x_n \text{ in } \mathbb{R}^n. \]

- E.g. \( f(x_1, x_2) = x_1 + \frac{x_2}{x_1} \); 2-digit decimal arith; \( x_1 = [3, 4] \), \( x_2 = [3, 5] \):

  \[
  x_1 + \frac{x_2}{x_1} = [3, 4] + \left[ \frac{3, 5}{3, 4} \right] = [3, 4] + \left[ \frac{3}{4}, \frac{5}{3} \right] \xrightarrow{\text{round}} [3, 4] + [.75, 1.7] \\
  = [3.75, 5.7] \xrightarrow{\text{round}} [3.7, 5.7] = f(x_1, x_2) = y.
  \]

  \( y \) does contain the range of \( f \) over \( [3, 4] \times [3, 5] \), which is \( [4, 5\frac{1}{4}] \).
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Why do intervals need new algorithms?

Example: Newton’s method for solving a 1-D nonlinear system.

Why a specific iteration for the interval case? Usual formula:

\[ x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)} \]

Direct interval transposition:

\[ x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)} \quad (f, f' = \text{interval versions of } f, f', \text{ see last slide.}) \]

Width of the resulting interval:

\[ w(x_{k+1}) = w(x_k) + w\left(\frac{f(x_k)}{f'(x_k)}\right) > w(x_k) \]

Divergence!
Back to basic theory

Let $f$ be $C^1$ function on interval $I$.
By Mean Value Theorem MVT, $\forall$ root $z \in I$, $\forall x \in I$, $\exists \xi \in I$ s.t.

$$f(x) = f(x) - f(z) = (x - z)f'(\xi)$$  \hspace{1cm} (1)

so provided $f'(\xi) \neq 0$, see later,

$$z = x - \frac{f(x)}{f'(\xi)}.$$  \hspace{1cm} (2)

A searchlight shone from $(x, f(x))$ bounded by lowest & highest slopes of $f$ on $I$ is certain to illuminate any root $z = (z, 0)$ in $I$. 

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Computable version

When computing $x - \frac{f(x)}{f'(\xi)}$

- $x$ is “point”. Arbitrary in $I$ ($\forall$), typically midpoint.
- $f(x)$ must be “interval”, as $f$ is code, liable to roundoff.
- $f'(\xi)$ must be “interval”, as (a) $f'$ is code, (b) $\xi \in I$ is uncertain ($\exists$).

So ($\forall$) if any root $z \in I$ then also

$$z \in \left( x - \frac{\lfloor f(x) \rfloor}{\lfloor f'(\xi) \rfloor} \right)$$

[...] meaning “some interval containing”

or in more current notation, renaming $I$ as $x$

$$z \in \left( x - \frac{f([x])}{f'(x)} \right) = \left( \text{point} - \frac{\text{interval function of point}}{\text{interval function of interval}} \right)$$

where $[x]$ is 1-point interval $\{x\}$ and $f, f'$ are interval versions of $f, f'$. 
More general picture

- Actually searchlight shines in both directions, crucial when range of slopes includes + and − values:

\[ y = f(x) \]

- ...provided one interprets \( \div \) as reverse multiplication

\[ c / b = \text{(any solution of } bx = c), \quad \text{P1788's } \text{mulRev}(b,c). \]

So 0/0 means “whole real line” instead of “undefined”.
- Now we enclose all roots even when many exist! Note searchlight can split \( I \) into 2 pieces.
Interval Newton iteration

(Hansen–Greenberg 1983; Kearfott & many others since)

Set $x_0 = \text{initial interval } I$

For $k = 0, 1, 2, \ldots$

$x_k = \text{some chosen point in } x_k$

$Y_{k+1} = x_k - f([x_k])/f'(x_k)$ \quad (\div \text{ means } \text{mulRev})

$x_{k+1} = Y_{k+1} \cap x_k$

$Y_{k+1}$ can be union of 2 intervals
On the algorithm

This method guarantees to enclose all roots, but “2-way searchlight” case splits $x_k$ in two, producing a possible tree of computations.

Features, assuming $f$ is $C^1$ on initial interval:

- $x_{k+1} = \emptyset$ guarantees \( \nexists \) root in $x_k$.
- If $0 \notin f'(x_k)$, \( \exists \) at most one root in $x_k$ (which must be in $x_{k+1}$).
- Less obvious, if $Y_{k+1}$ is $\neq \emptyset$, bounded, $\subseteq x_k$, \( \exists \) just one root in $x_k$.  
On the algorithm
This method guarantees to enclose all roots, but “2-way searchlight” case splits $x_k$ in two, producing a possible tree of computations.
Features, assuming $f$ is $C^1$ on initial interval:
- $x_{k+1} = \emptyset$ guarantees $\not\exists$ root in $x_k$.
- If $0 \not\in f'(x_k)$, $\exists$ at most one root in $x_k$ (which must be in $x_{k+1}$).
- Less obvious, if $Y_{k+1}$ is $\neq \emptyset$, bounded, $\subseteq x_k$, $\exists$ just one root in $x_k$.

What does it show about the interval mindset?
- This analysis wasn’t rocket science, just a careful look at the $\forall$, $\exists$ in a use of the MVT.
- But in general, seeing how mathematics converts to interval algorithms takes time and practice.
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The need for a standard

- Dozens of excellent interval software packages have been written, with not quite compatible math foundations:
  - Support unbounded intervals and the empty set? Moore IA didn’t.
  - Is an interval a set of numbers? Kaucher IA has intervals like $[4, 3]$.
  - How to handle $\sqrt{[-2, 2]}$, or $x/y$ when $0 \in y$?

... as well as different software interfaces.

- Currently one can’t write algorithms that are portable at a mathematical level, let alone portable software.

- At Dagstuhl, Germany (Jan ’08) a project was started, which became IEEE Working Group P1788 “A standard for interval arithmetic”.

- Officers: chair, vice-chair, technical editor (me), co-editors, web master, secretary/archivist, voting tabulator. ~ 45 voting members.

- We have (May ’14) voted to approve a final document, and (Aug ’14) initiated IEEE “sponsor ballot” stage.
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Definition of an interval

- In the current standard
  - An interval \( x \) is a set of numbers.
  - \( \pm \infty \) not allowed as members of \( x \), so intervals are subsets of \( \mathbb{R} \).
  - Open/half-open intervals not allowed, but unbounded intervals are.
  - Empty set is an interval.

So interval means topologically closed and connected subset of \( \mathbb{R} \).

- There is a framework—so called *flavors*—to support alternative mathematical foundations, such as Kaucher IA in which an interval is an ordered pair \((x, \bar{x})\) with \( x, \bar{x} \in \mathbb{R} \):

\[
(x, \bar{x}) \text{ “means” } \begin{cases} 
\text{set } [x, \bar{x}] \subset \mathbb{R} & \text{if } x \leq \bar{x} \text{ (“proper” interval)} \\
\text{something weird} & \text{if } x > \bar{x} \text{ (“improper” interval)}
\end{cases}
\]
The Levels structure

Distinguish 4 specification levels (as in floating point standard IEEE754):

Level 1. Mathematical theory of intervals & their operations.

Level 2. Finite precision intervals—datums—& operations, independently of their representation.

Level 3. Representation of datums by objects, e.g. in terms of floating point numbers.

Level 4. Encoding of Level 3 objects as bit-strings.
Maps between levels are crucial—especially $L_1 \leftrightarrow L_2$. We decided:

- Each datum is a mathematical interval, i.e.
  \[ L_2 \text{ datums} \xrightarrow{\text{identity map}} L_1 \text{ intervals} \quad (*) \]

- Datums are organised into finite sets $\mathbb{T}$ called interval types.

- A $L_1$ interval $x$ maps to an interval of type $\mathbb{T}$ (a $\mathbb{T}$-interval) by the $\mathbb{T}$-hull operation = smallest (in $\supseteq$ sense) $\mathbb{T}$-interval that contains $x$.
  \[ L_1 \text{ intervals} \xrightarrow{\mathbb{T}\text{-hull}} L_2 \text{ datums of type } \mathbb{T} \quad (**) \]

- To do an operation $x \bullet y$ at $L_2$ on $\mathbb{T}$-intervals:
  map $x, y$ to $L_1$ by $(*)$; do operation at $L_1$; map back to $L_2$ by $(**)$. 

Looks trivial but isn’t! Not all IA theories are clear on this.
IMO, this choice defines the whole character of the standard.
Then two obvious rules

- **L2 \leftrightarrow L3**: Each L2 datum is represented by at least one L3 object; each L3 object represents at most one L2 datum.
- **L3 \leftrightarrow L4**: Each L3 object is encoded by at least one L4 bitstring; each L4 bitstring encodes at most one L3 object.
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Exception handling—a hypothetical scenario

Less than 10 years hence in the Old Bailey . . .

- Crown vs Google concerns Google’s driverless car GDC. One of them badly injured a pedestrian who stepped into the road in front of it.
- GDC’s emergency stop system is designed to act faster than a good human driver (undisputed) but is it badly implemented (disputed)?
- The software uses an interval algorithm, built on a 1788-conforming library, which applies Brouwer’s fixed point theorem.
- Depending on what software bugs are found (if any), liability might lie with the pedestrian’s negligence? GDC’s software implementers? the 1788 library implementers? 1788’s mathematicians? etc.
- A lot of ££ rides on whether 1788-based code might be wrong, when deciding that a function is defined & continuous on a box.
Exception handling—context

The basic problem is how (at Level 1) to treat operations that aren’t everywhere defined [and/or continuous] on the input box, e.g.

\[
\begin{align*}
\text{(real) square root } & \sqrt{[-2, 2]}; & \frac{[2, 3]}{[-1, 1]}; & \text{floor}([2.5, 4.5])
\end{align*}
\]

- We decided the default is “evaluate where defined, ignore where undefined”, called non-stop or loose evaluation, e.g.

\[
\sqrt{[-2, 2]} = \{ \sqrt{x} \mid x \in [-2, 2] \text{ and } x \geq 0 \} = [0, \sqrt{2}]
\]

with no error reported. (Like IEEE754 floating point.) OK for, e.g., many global optimisation methods.

- Not OK for applying Brouwer’s theorem, which needs to know a function is everywhere defined & continuous on a box.

- Also not OK for some graphics rendering algorithms, which need to know definedness, not bothered about continuity.
So one needs a mechanism to track whether a library operation has these desirable properties of definedness and/or continuity.

This leads to a powerful extension of the Fundamental Theorem of IA based on theorems of set theory & analysis:

- If for function $f$ given by an expression, each individual library operation is everywhere defined on its inputs, then the same goes for $f$.
- Same with defined replaced by defined & continuous.

We rejected the IEEE754 FP standard’s method of *global flags*—obsolete for today’s massively parallel platforms.

Instead provide facility of decorated interval $(y, dy) = \text{interval } y \text{ plus tag } dy$ (a decoration)$^1$ giving information about definedness, continuity, etc.

\[ ^1 \text{dy just means “decoration for } y \text{”, nothing to do with differentials!} \]
Formally, a decoration $d$ is a label for an assertion (predicate) $p_d(f, x)$ about a function $f : \mathbb{R}^n \rightarrow \mathbb{R}$ and a box $x \subseteq \mathbb{R}^n$, for arbitrary $n$.

- **5 decorations** are defined in increasing order of “goodness”
  ill $< \text{trv} < \text{def} < \text{dac} < \text{com}:
  - ill: Label for ill-formed intervals, formally “$f$ is nowhere defined”.
  - trv: (trivial) Always true = “no information”.
  - def: $f$ is everywhere defined on $x$.
  - dac: As def, plus everywhere continuous on $x$.
  - com: As dac, plus bounded at Level 2 (no overflow while computing it).

- Let $(y, dy)$ result from evaluating arithmetic expression $f(x_1, \ldots, x_n)$ on correctly initialised decorated intervals $(x_1, dx_1), \ldots, (x_n, dx_n)$.

- Then, in addition to $y \supseteq \text{range of } f \text{ over } x = x_1 \times \ldots \times x_n$, the decoration $dy$ makes a true assertion about $f$ over $x$.
  E.g. if $dy = \text{def}$ then $f$ was proved to be everywhere defined on $x$. 
This exception handling method is the feature that most distinguishes 1788 from earlier IA systems.

There’s no magic: it relies on systematically exploiting facts such as “composition of everywhere defined functions is everywhere defined”. An Annex in the Standard contains a rigorous proof of correctness of the decoration system: a **Fundamental Theorem of Decorated Interval Arithmetic**.

Like range enclosures, it’s often *not sharp*, e.g. may return `trv` (no info) or `def` (defined) when actually `dac` (defined & continuous) is true.

Much of the craft of IA is knowing how to “sharpen” such info, e.g. by cutting an input box into smaller boxes handled separately.
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Difficulties the group encountered

Certain topics caused heated debate. Examples:

- Choice of foundational math model of intervals & operations. We split into “set-based” (mostly academic) and “Kaucher” (earn $$ from intervals) factions.
- Flavors: the way of accommodating different foundations.
- The decoration scheme—result of over a year’s discussion.
- Correctness proof—to use in hypothetical litigation above?
- Kinds of exception to which decorations are unsuited, e.g. bad interval constructor calls.
- What to say about accuracy? Just leave it as a QoI issue?
- Exact dot-product—should it be part of the 1788 standard?
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The current document has

~ 60 pp of main text (requirements) of which roughly 50% Level 1, 45% Level 2, 5% Level 3, a tiny bit of Level 4.

~ 15 pp of operation tables and other help for implementers

~ 18 pp of the *Basic Standard*, a cut down, simpler to implement, version.

Vote of the group approved it in May 2014.

We are preparing *Sponsor Ballot* stage of IEEE process, where it is examined by a selected group intended to be

– representative of academia, software developers, industry, etc.;

– geographically balanced.

This should result in changes, hopefully minor . . .

and we hope it will be accepted as an IEEE document in early 2015.
[\_] good

[\_] better

[\_] even better

dac
Decorations example

- Consider fix point problem \( g(x) = x \) where
  \[
g(x) = 2\sqrt{x} - \frac{1}{2}.
  \]
  Roots are \( x = \frac{3}{2} \pm \sqrt{2} = 0.0858 \ldots \) or \( 2.9142 \ldots \)

- Use fixed point iteration \( x_{n+1} = g(x_n) \)

- Initial \( x_0 = [2, 3] \) gives
  \[
x_1 = [2\sqrt{2} - \frac{1}{2}, 2\sqrt{3} - \frac{1}{2}] = [2.3 \ldots, 2.9 \ldots] \subset x_0.
  \]
  This is genuine and (Brouwer) shows a fixpoint exists in \( x_1 \).

- Initial \( x_0 = [-1, \frac{1}{16}] \) gives
  \[
x_1 = 2\sqrt{[-1, \frac{1}{16}]} - \frac{1}{2} = 2[0, \frac{1}{4}] - \frac{1}{2} = [0, \frac{1}{2}] - \frac{1}{2} = [-\frac{1}{2}, 0], \text{ again } \subset x_0!
  \]
  This is spurious, due to 1788 (undecorated) arithmetic discarding the negative part of \( x_0 \) without comment.
Using decorated interval arithmetic—using the rules for propagating decorations through operations, which I skate over—the 2nd example gives

\[
x_1 = [2]_{dac} \times \sqrt{[-1, \frac{1}{16}]_{dac} - [\frac{1}{2}]_{dac}}
\]

\[
= [2]_{dac} \times [0, \frac{1}{4}]_{trv} - [\frac{1}{2}]_{dac}, \quad \text{trv} = "no information"
\]

\[
= [0, \frac{1}{2}]_{trv} - \frac{1}{2}_{dac}
\]

\[
= [-\frac{1}{2}, 0]_{trv},
\]

while the 1st example produces

\[
x_1 = [2.3\ldots, 2.9\ldots]_{dac}.
\]

I.e. in 1st case we conclude conditions of Brouwer’s Theorem are satisfied, but in 2nd case are unable to do so.