Programming Techniques for Exact Real Arithmetic

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(joint work with Ivo List & Paul Taylor)

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- Further material: `http://www.paultaylor.eu/ASD/`

Axioms for real numbers

The real numbers $\mathbb{R}$ are:

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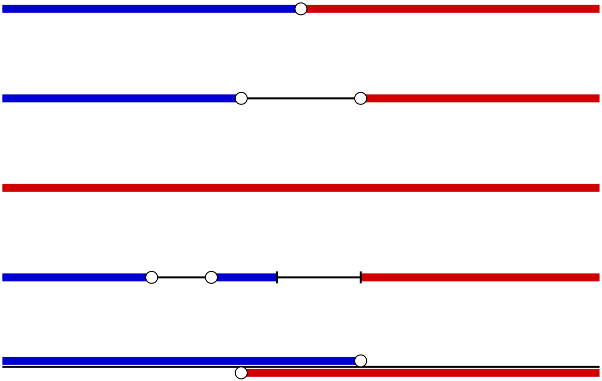
- and $[a, b]$ is compact ("can verify something holds")

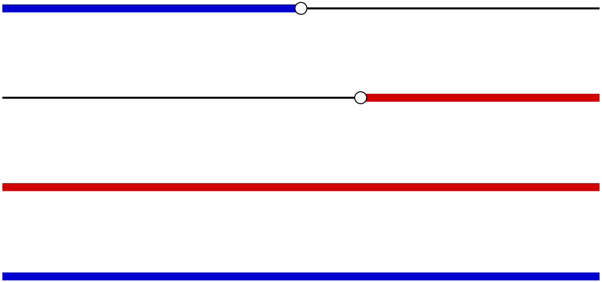
Dedekind cuts

A *cut* is a pair of *rounded*, *bounded*, *disjoint*, and *located* open sets.



Lower and upper reals

By taking the lower rounded sets we obtain the *lower reals*, and similarly for *upper reals*. These are more fundamental than reals.



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- A number a determines a cut, which determines a:

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- The full notation for cuts is

$$\mathsf{cut}\ x : [a, b]\ \mathsf{left}\ \phi(x)\ \mathsf{right}\ \psi(x)$$

  This means that the cut determines a number in $[a, b]$.

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  - universal quantifier: $\quad \forall x : [a, b]$

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- We use this to express topological and analytic notions *logically*.

Example: $\mathbb{R}$ is locally compact

- Classically: for open $U \subseteq \mathbb{R}$ and $x \in \mathbb{R}$,

$$x \in U \iff \exists\, d, u \in \mathbb{Q} \,.\, x \in (d, u) \subseteq [d, u] \subseteq U$$

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- Topologically: for $\phi : \mathbb{R} \to \Sigma$ and $x : \mathbb{R}$,

$$\phi(x) \iff \exists\, d, u \in \mathbb{Q} \,.\, d < x < u \wedge \forall\, y \in [d, u] \,.\, \phi(y)$$

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$$U \cap V = \emptyset \quad \text{and} \quad U \cup V = [0,1]$$

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- Topologically: for $\phi, \psi : [0,1] \to \Sigma$, if

$$\begin{aligned} (\exists\, x \in [0,1] \,.\, \phi(x) \wedge \psi(x)) &\iff \bot \quad \text{and} \\ (\forall\, x \in [0,1] \,.\, \phi(x) \vee \psi(x)) &\iff \top \end{aligned}$$

then $(\forall\, x \in [0,1] \,.\, \phi(x)) \vee (\forall\, x \in [0,1] \,.\, \psi(x))$.

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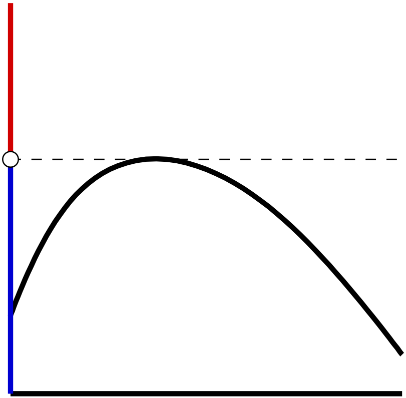
- Note: A must be *overt* and B *compact*.

The maximum of $f : [0,1] \to \mathbb{R}$



cut x left $(\exists\, y \in [0,1]\,.\, x < f(y))$
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Cauchy completeness

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- Its limit is the cut

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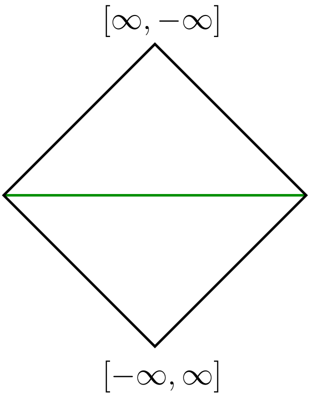
- Not surprisingly, we compute with (improper) intervals.

The interval lattice L

- The lattice of **pairs** $[a, b]$, where a is *upper* and b *lower real*.



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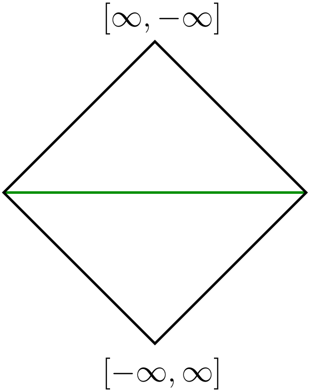

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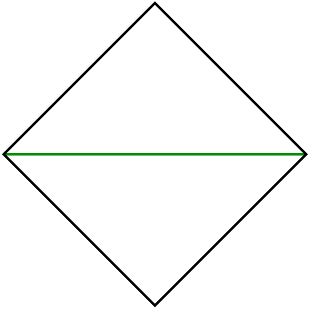
- The lattice contains $\mathbb{R}$ as $[a, a]$.

$[\infty, -\infty]$



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- We also extend $<$ to $L \times L \to \Sigma$:

$$[a, b] < [c, d] \iff b < c$$

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- Easy cases:

$$\begin{aligned} \bot^- &= \bot & \bot^+ &= \bot \\ \top^- &= \top & \top^+ &= \top \\ (\phi \wedge \psi)^- &= \phi^- \wedge \psi^- & (\phi \wedge \psi)^+ &= \phi^+ \wedge \psi^+ \\ (\phi \vee \psi)^- &= \phi^- \vee \psi^- & (\phi \vee \psi)^+ &= \phi^+ \vee \psi^+ \\ (e_1 < e_2)^- &= (e_1^- < e_2^-) & (e_1 < e_2)^+ &= (e_1^+ < e_2^+). \end{aligned}$$

Approximants for cuts and quantifiers

- Cuts:

$$(\mathsf{cut}\ x : [a,b]\ \mathsf{left}\ \phi(x)\ \mathsf{right}\ \psi(x))^{-} = [a,b]$$
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- Quantifiers:

$$\phi([a,b]) \implies \forall\, x \in [a,b]\,.\,\phi(x) \implies \phi(\tfrac{a+b}{2})$$

$$\phi(\tfrac{a+b}{2}) \implies \exists\, x \in [a,b]\,.\,\phi(x) \implies \phi([b,a])$$

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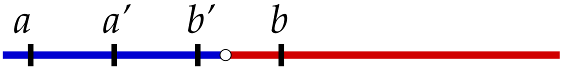
- This amounts to searching with *bisection*.

Refinement of cuts

- To refine a cut

$$\text{cut } x : [a, b] \text{ left } \phi(x) \text{ right } \psi(x)$$

we try to move $a \mapsto a'$ and $b \mapsto b'$.



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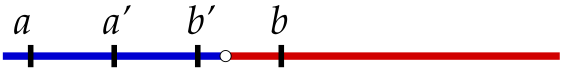
$$\text{cut } x : [a, b] \text{ left } \phi(x) \text{ right } \psi(x)$$

we try to move $a \mapsto a'$ and $b \mapsto b'$.



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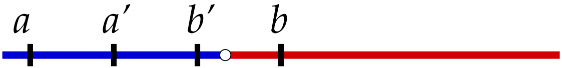

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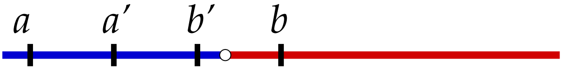

- If $\phi^-(a') = \top$ then move $a \mapsto a'$.
- If $\psi^-(b') = \top$ then move $b \mapsto b'$.
- One or the other endpoint moves eventually because cuts are located.

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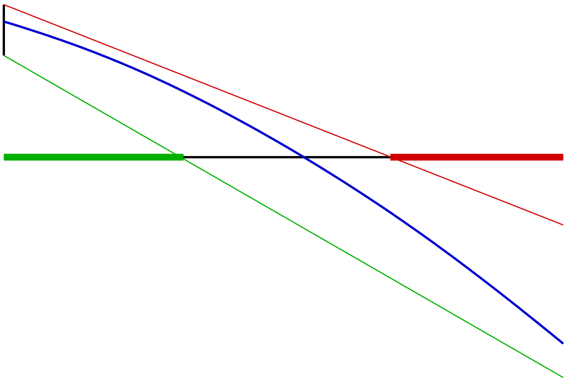
- Evaluation may not terminate, but this is expected, as ϕ is only *semi*decidable.

Speeding up the computation

Estimate an inequality $f(x) < 0$ on $[a, b]$ by approximating f with a linear map from above and below.



This is essentially Newton's interval method.

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- How do we incorporate $\mathbb{N}$ and recursion?
- How to extend Newton's method to improper intervals?
- How to extend Newton's method to the multivariate case?
- Can we do higher-type computations $\int$ and $\frac{d}{dx}$?
- Can this lead to a useful domain-specific language?