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Lehrstuhl für
INFORMATIK I
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Cutting Convex Polyhedra by Planes

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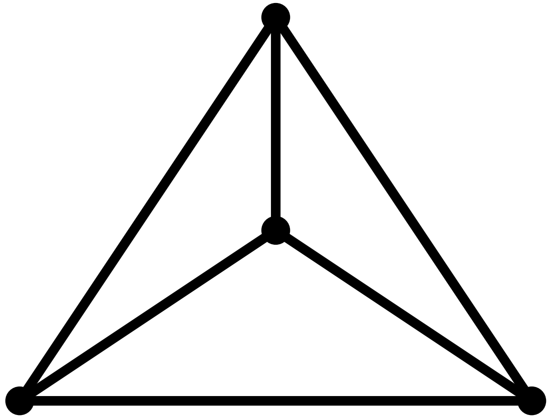
Polyhedral graphs

Steinitz's theorem says that a graph G is isomorphic to the 1-skeleton of a three-dimensional convex polyhedron if and only if G is planar and 3-connected. By this reason 3-connected planar graphs are called *polyhedral*.



Cutting planar graphs by lines

Let π be a drawing of a graph G and ℓ be a line. We say that ℓ *crosses* an edge or a face of π if ℓ intersects it at an inner point.

Denote the number of edges (resp. faces) of π that ℓ crosses by $\bar{e}(\pi, \ell)$ (resp. $\bar{f}(\pi, \ell)$).

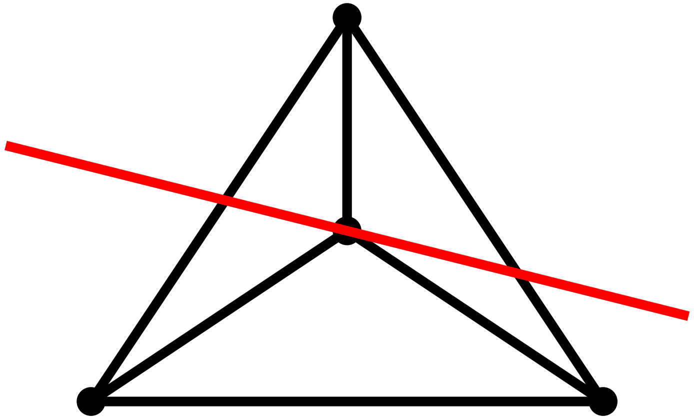
$$\bar{e}(G) = \max_{\pi, \ell} \bar{e}(G, \pi)$$

$$\bar{f}(G) = \max_{\pi, \ell} \bar{f}(G, \pi)$$

Denote the number of vertices of π on ℓ by $\bar{v}(\pi, \ell)$.

$$\bar{v}(G) = \max_{\pi, \ell} \bar{v}(\pi, \ell)$$

Example



$$\bar{v}(\pi, \ell) = 1 \ \bar{e}(\pi, \ell) = 2, \ \bar{f}(\pi, \ell) = 3.$$

Some Theorems

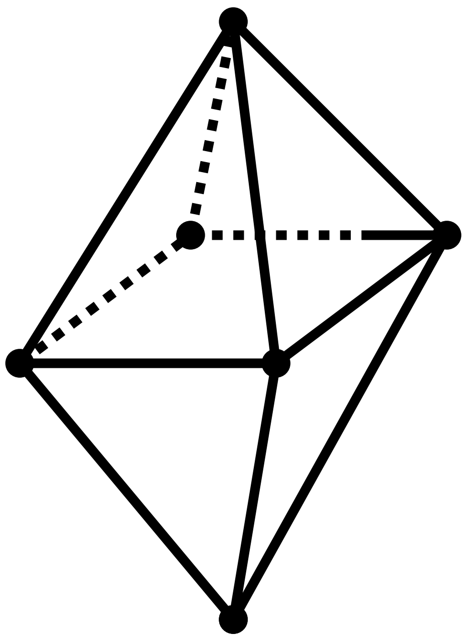
For every triangulation T, $\bar{v}(T) \leq \bar{f}(T) = \bar{e}(T) \leq c(T^*)$.

(The *circumference* $c(G)$ of a graph G is the length of a longest cycle in G and G^* is the dual of a polyhedral graph G).

If G be a planar graph such that degree of each vertex of G is at least k then $\bar{e}(G) \geq (k/2 - 1)\bar{v}(G)$. In particular, $\bar{e}(G) \geq (1/2)\bar{v}(G)$ for each polyhedral graph G.

Cutting convex polyhedra by planes

Let G be a polyhedral graph, π be a convex polyhedron whose 1-skeleton is isomorphic to G, and ℓ be a plane. Values $\overline{\overline{e}}(G)$, $\overline{\overline{f}}(G)$, and $\overline{\overline{v}}(G)$ are defined similarly to $\bar{e}(G)$, $\bar{f}(G)$, and $\bar{v}(G)$ from the preceding section.



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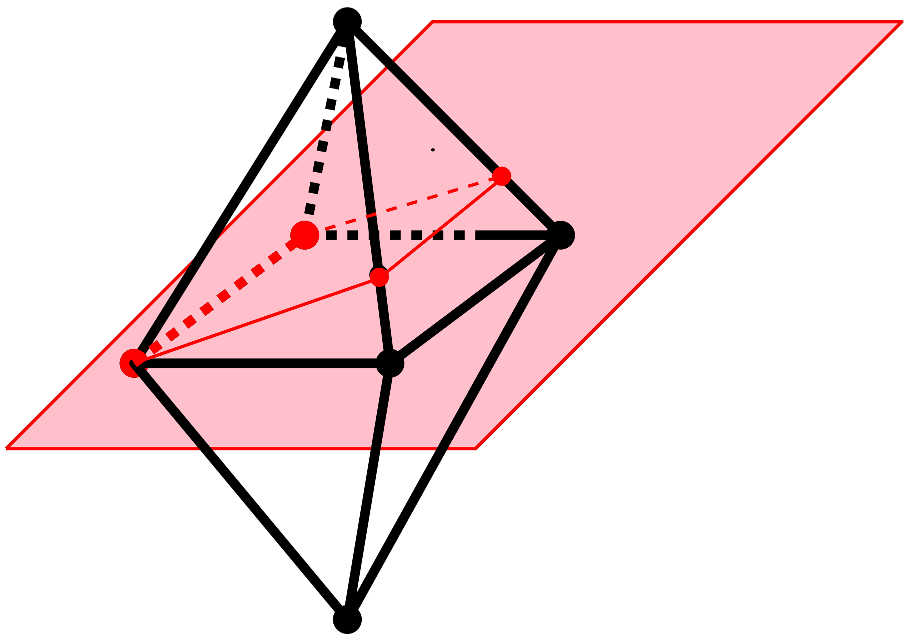

$$\overline{\overline{v}}(\pi, \ell) = 2,\ \overline{\overline{e}}(\pi, \ell) = 3,\ \overline{\overline{f}}(\pi, \ell) = 3.$$

More Theorems

For every polyhedral graph G, $\overline{\overline{v}}(G) \leq \overline{\overline{f}}(G) = \overline{\overline{e}}(G) \leq c(G^*)$.

The last inequality was used by Grünbaum to show that for some G we have $\overline{\overline{e}}(G) = O(n^{\alpha})$ for some $\alpha < 1$.

If G be a planar graph such that degree of each vertex of G is at least k then $\overline{\overline{e}}(G) \geq (k/2 - 1)\,\overline{\overline{v}}(G)$.

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There are polyhedral graphs G on n vertices with $\overline{\overline{v}}(G) > (2/3)n - 2$ and $c(G) = O(n^{\log_3 2})$.

Polyhedra with small planar sets of vertices

The *shortness exponent* of a class of graphs $\mathcal{G}$ is the limit inferior of quotients $\log c(G)/\log v(G)$ over all $G \in \mathcal{G}$. Let σ denote the shortness exponent for the class of cubic polyhedral graphs. It is known that

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Proof. Let $V(G) = \{v_1, \ldots, v_n\}$, $\chi : V(G) \to \{1, \ldots, \chi(G)\}$ be a coloring of G, and $x_1, \ldots, x_n$ be real numbers which are linearly independent over the field $\mathbb{Q}$. Then $d(v_i) = (x_i, \chi(v_i))$ is the reqired drawing.

Drawing graphs on several planes

$\pi(G)$ is equal to the smallest size r of a partition $V(G) = V_1 \cup \ldots \cup V_r$ such that every V_i induces a planar subgraph of G. Therefore,

$$\frac{1}{4}\chi(G) \leq \pi(G) \leq \chi(G)$$

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Compute $\rho(K_n)$.