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Interdeparture Time Correlations of the Discrete-time GI/GI/1 Queue

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Abstract

In this paper, we present a method to compute the coefficient of correlation for two consecutive interdeparture time periods of a discrete-time $GI/GI/1$ system. In addition, we provide an algorithm to create the one-step transition probability matrix of the interdeparture time process of this system. The presented methodology can be applied to approximate the correlations of continuous-time $GI/GI/1$ systems with a high degree of accuracy.

1 Introduction

For assessing the performance of tandem queues or, in a more general context, queueing networks with general interarrival and service times, a suitable model for the interdeparture times is required. Normally, only the first few moments or, in some cases an approximation of the distribution of the interdeparture times of a system are taken into consideration due to complexity reasons. To increase the degree of feasibility of a performance evaluation, independence of the interdeparture times is assumed. In general, however, departures are correlated and performance measures based on an uncorrelated stream of departures can be misleading. A study by Livny et al. [7] shows the tremendous effect of correlated arrivals on the waiting time of the customers.

Despite its importance, the evaluation of the coefficients of correlation of interdeparture times of $GI/GI/1$ systems is not addressed in a general manner in the literature. Nevertheless, there are several solutions of diverse complexity for a limited number of interarrival time and service time distributions.

In this paper, we provide a method to compute the coefficients of correlation of two consecutive interdeparture times of a discrete-time $GI/GI/1$ system with arbitrary interarrival time and service time distributions. The comprehensiveness of the approach is supported by the fact that the complete derivation takes place in time domain.

The approach can be used to approximate the coefficients of correlation of continuous-time systems with a high degree of accuracy. In addition, we provide the one-step transition probability matrix of the interdeparture time process that may serve as a model for the arrivals of the consecutive system.

The paper is organized as follows. In Section 2, the notation and the concepts of discrete-time analysis in time domain are presented. This section contains the methodology to compute the distributions of waiting times and interdeparture times of a discrete-time $GI/GI/1$ system. In Section 3, the computation of the coefficient of correlation of the interdeparture times is derived. The algorithm to estimate the one-step transition probability matrix of the interdeparture time process is provided in Section 4. Information about the approximation of continuous-time systems by the presented approach and the accuracy of that method can be obtained from Section 5.

2 Discrete-time Analysis Basics

The subject of our study is the discrete-time $GI/GI/1$ queuing system with infinite waiting room [1]. The time interval between the consecutive arrivals of two customers is described in terms of a discrete probability mass function (*PMF*) $a_n(k)$: $a_n(k)$ is the probability of having an interval of an integer number of k time units between the arrival of customer number n and customer number $n+1$. The service time of customer n is given in terms of a discrete probability mass function $b_n(k)$. The interarrival times and the service times are i.i.d. random variables. There is one single server and an infinite waiting room. The customers are served in the order of their arrival (first-come-first-served, FCFS).

The sum of the service times of the customers waiting for service and the remaining service time of the customer currently being served is termed the *unfinished work*. The service time distribution is discrete, and so is the distribution of the unfinished work. We denote the PMF of the unfinished work by $u(k)$. At the instant of arrival of customer n the unfinished work is increased by the service time of customer n. If the server is busy the unfinished work is decreased by one (discrete) work unit per (discrete) time unit.

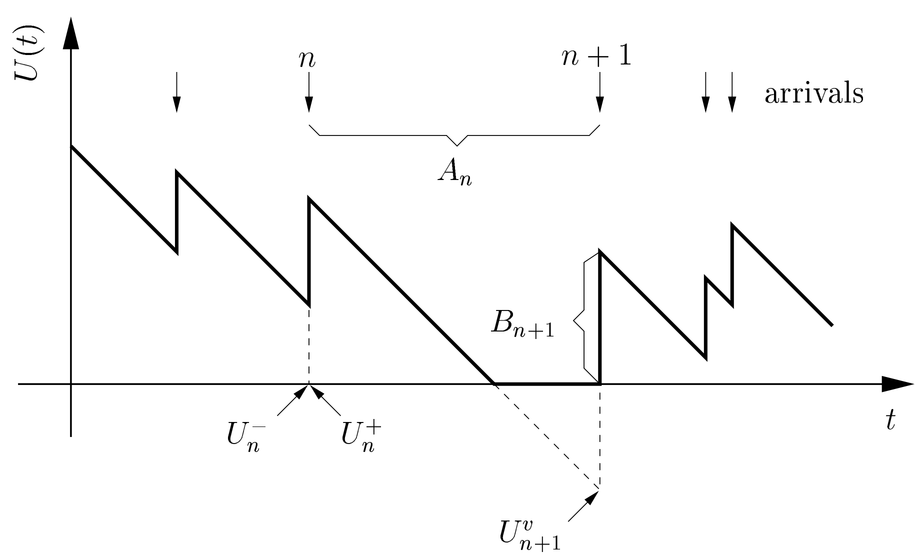

Figure 1: Unfinished work process of the $GI/GI/1$ system.

In the following the random variables of the interarrival time, service time etc. are denoted by uppercase letters corresponding to the lowercase letters of the PMF's, e.g. B_n denotes the service time random variable of customer n. Indicating '*just prior to the arrival instant*' by adding a superscript '$-$' and '*just after the arrival instant*' by adding a superscript '+' to the random variables and PMF's respectively, we obtain the following relations for the development of the unfinished work U:

$$U_n^+ = U_n^- + B_n, \tag{1}$$

$$U_{n+1}^- = \max\{U_n^+ - A_n, 0\}. \tag{2}$$

The previous equation results from the fact that the unfinished work is decreased by one work unit per time unit, i.e. during A_n time units U_n is diminished by A_n work units. Combining the previous two equations the following recursive equation is obtained:

$$U_{n+1}^- = \max\{U_n^- + B_n - A_n, 0\}. \tag{3}$$

Turning from random variables to the corresponding PMF's this equation becomes:

$$u_{n+1}^-(k) = \pi_0[u_n^-(k) \circledast b_n(k) \circledast a_n(-k)]. \tag{4}$$

Here, the linear operator $\pi_0[\cdot]$ "sweep[s] the probability in the negative half-line up to the origin" ([6], Ch. 2.6):

$$\pi_0[z(k)] = \begin{cases} 0 & \text{for } k < 0, \\ \sum_{i=-\infty}^{0} z(i) & \text{for } k = 0, \\ z(k) & \text{for } k > 0. \end{cases} \tag{5}$$

The operator $\circledast$ denotes the discrete convolution. Defining the *system function* $c_n(k)$ by the cross-correlation of $a_n(k)$ and $b_n(k)$, i.e.

$$c_n(k) = b_n(k) \circledast a_n(-k), \tag{6}$$

we obtain

$$u_{n+1}^-(k) = \pi_0\left[u_n^-(k) \circledast c_n(k)\right]. \tag{7}$$

Observing the process of the unfinished work at arrival instants only, the unfinished work just prior to the arrival instant of a customer is equal to this customer's waiting time [5].

Denoting the (discrete) PMF of the waiting time distribution of customer number n by $w_n(k)$ we obtain the final result:

$$w_{n+1}(k) \quad = \quad \pi_0 \left[w_n(k) \circledast c_n(k) \right]. \tag{8}$$

The recursive formula just obtained can be viewed as the program to compute the waiting time distribution of the $GI/GI/1$ queuing system iteratively.

The recursive scheme describes the *non-stationary behavior* of the system even with interarrival time and service time PMF's changing on a customer's basis. The *equilibrium* distribution is obtained by iterating until convergence is achieved, to within an appropriate criterion.

In the following, we will outline the the computation of the PMF's of the idle period $i(k)$ and the interdeparture time $d(k)$ [8]. Since the service time and the amount of unfinished work are principally interchangeable the length of an idle period is exactly the amount of virtual unfinished work $u^v(k)$ with negative values, where

$$u^v(k) \quad = \quad w(k) \circledast a(-k). \tag{9}$$

We obtain the PMF of the idle period $i(k)$ by

$$i(k) \quad = \quad \Pr\big\{I = k\big\} \quad = \quad \Pr\big\{U^v = -k\big\} \quad = \quad K \cdot u^v(-k). \tag{10}$$

where the normalization constant K is given by

$$K^{-1} = \sum_{j=1}^{\infty} u^v(-j). \tag{11}$$

Note, that I is defined to be non-zero.

The probability p_{idle} to leave an empty system is

$$p_{\text{idle}} \quad = \quad \frac{EA - EB}{EI}. \tag{12}$$

Hence, the interdeparture time PMF $d(k)$ is obtained by

$$d(k) \quad = \quad p_{\text{idle}}\big(i(k) \circledast b(k)\big) + \big(1 - p_{\text{idle}}\big) b(k). \tag{13}$$

The above algorithms can also be used to approximate the *continuous*-time $GI/GI/1$ queuing system. To do this the continuous-time distribution functions are approximated by suitably chosen discrete ones.

3 Computation of the Coefficient of Correlation

This section is devoted to the computation of the lag-1 coefficient of correlation of the interdeparture times of the discrete-time system introduced in the above section. The coefficient of correlation $\mathrm{Corr}[D_n\, D_{n+1}]$ is defined by

$$\mathrm{Corr}[D_n\, D_{n+1}] = \frac{\mathrm{E}[D_n D_{n+1}] - \mathrm{E}[D]^2}{\mathrm{VAR}[D]} \tag{14}$$

where the mean $\mathrm{E}[D]$ and the variance $\mathrm{VAR}[D]$ of the interdeparture times can be obtained directly from PMF $d(k)$ (see Equation (13)). D_n denotes the interdeparture time between customer n and customer $n+1$

In the sequel, we derive the computation of $\mathrm{E}[D_n D_{n+1}]$. Given A_n, B_n, B_{n+1}, and U_n^v, the system evolves according to the following four cases.

Case 1: Busy period at arrival of customer n ($U_n^v \geq 0$) and busy period at arrival of customer $n+1$ ($A_n \leq B_n + U_n^v$). This results in $D_n = B_n$ and $D_{n+1} = B_{n+1}$ (cf. Figure 2).

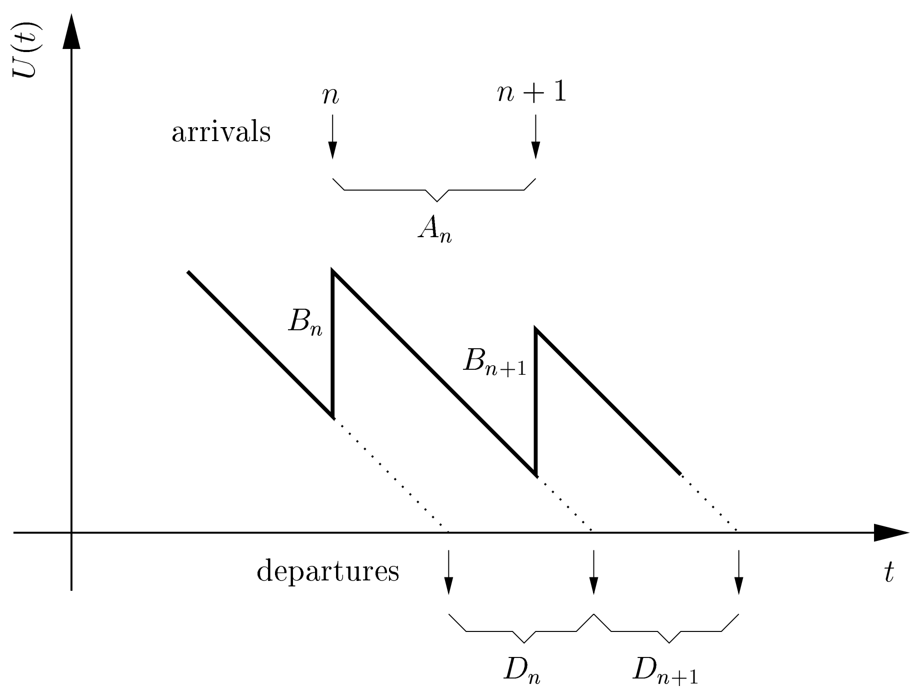

Figure 2: System evolution: case 1.

Case 2: Busy period at arrival of customer n ($U_n^v \geq 0$) and idle period at arrival of customer $n+1$ ($A_n > B_n + U_n^v$). This results in $D_n = B_n$ and $D_{n+1} = B_{n+1} + A_n - U_n^v - B_n$ (cf. Figure 3).

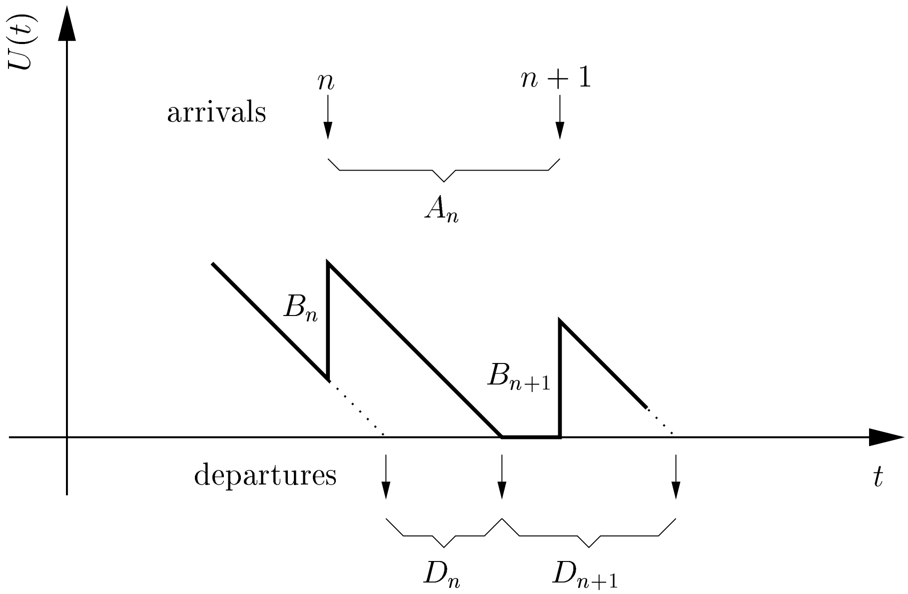

Figure 3: System evolution: case 2.

Case 3: Idle period at arrival of customer n ($U_n^v < 0$) and busy period at arrival of customer $n+1$ ($A_n \leq B_n$). This results in $D_n = B_n - U_n^v$ and $D_{n+1} = B_{n+1}$ (cf. Figure 4).

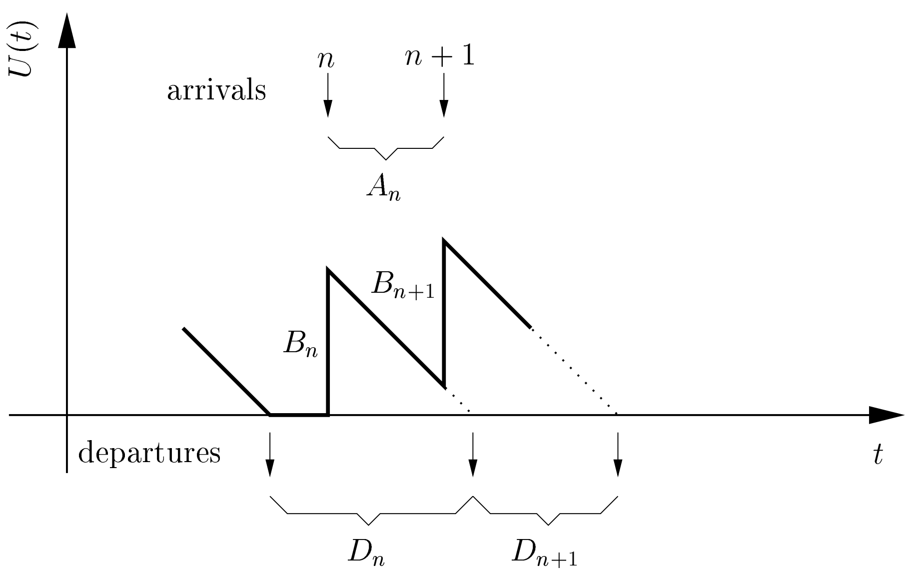

Figure 4: System evolution: case 3.

Case 4: Idle period at arrival of customer n ($U_n^v < 0$) and idle period at arrival of customer $n+1$ ($A_n > B_n$). This results in $D_n = B_n - U_n^v$ and $D_{n+1} = B_{n+1} + A_n - B_n$ (cf. Figure 5).

The value of $\mathrm{E}[D_n D_{n+1}]$ is now obtained by summing over the enumeration of all possible

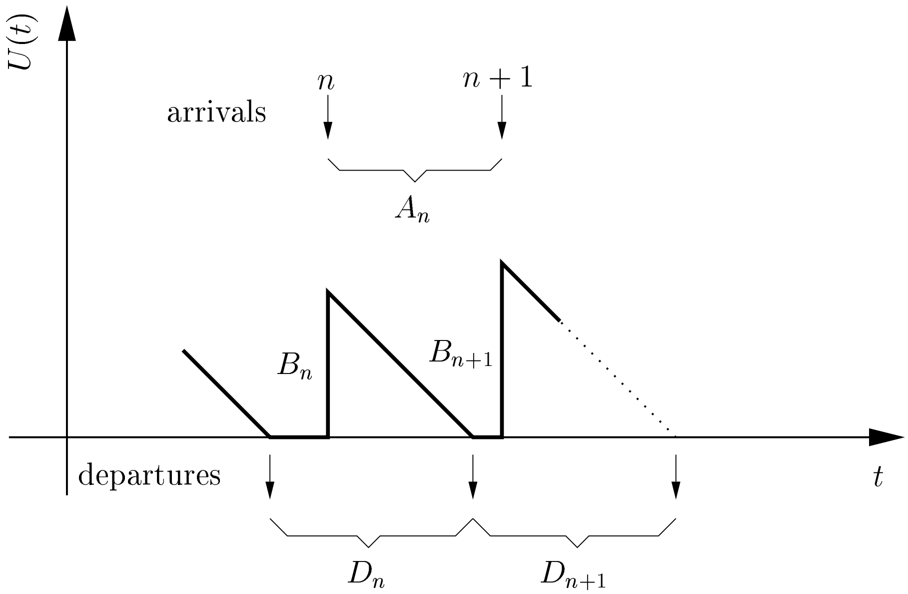

Figure 5: System evolution: case 4.

realizations of A_n, B_n, B_{n+1}, and U_n^v weighted by respective probabilities and the resulting values of D_n and D_{n+1}.

$$
\begin{aligned}
\mathrm{E}[D_n D_{n+1}] \quad = \quad & \sum_{i_{B_n}=0}^{\infty} \quad \sum_{i_{B_{n+1}}=0}^{\infty} b(i_{B_n})\, b(i_{B_{n+1}}) \\
& \left\{ \sum_{i_U=0}^{\infty} i_{B_n}\, u^v(i_U) \left[\underbrace{i_{B_{n+1}} \sum_{i_A=0}^{i_{B_n}+i_U} a(i_A)}_{\text{case 1}} + \right.\right. \\
& \left. + \underbrace{\sum_{i_A=i_{B_n}+i_U+1}^{\infty} (i_{B_{n+1}} + i_A - i_U - i_{B_n})\, a(i_A)}_{\text{case 2}} \right] + \qquad (15) \\
& + \sum_{i_U=-1}^{-\infty} (i_{B_n} - i_U)\, u^v(i_U) \left[\underbrace{i_{B_{n+1}} \sum_{i_A=0}^{i_{B_n}} a(i_A)}_{\text{case 3}} + \right. \\
& \left.\left. + \underbrace{\sum_{i_A=i_{B_n}+1}^{\infty} (i_{B_{n+1}} + i_A + -i_{B_n})\, a(i_A)}_{\text{case 4}} \right] \right\}.
\end{aligned}
$$

4 Computation of the Transition Matrix

To facilitate the approximation of the interdeparture process of the discrete-time GI/GI/1 system by its distribution and the lag-1 correlation, the one-step transition matrix $\mathbf{D}$ is often of greater value than $d(k)$ and $\mathrm{Corr}[D_n\, D_{n+1}]$ separately. The matrix $\mathbf{D}$ is defined by

$$\mathbf{D}_{rc} \;=\; \Pr\big\{D_{n+1} = c \,|\, D_n = r\big\}\,. \tag{16}$$

In principle, we have to deal with the four cases the computation of the coefficient of correlation $\mathrm{Corr}[D_n\, D_{n+1}]$ was based upon. The perspective, however, has to be changed. In Section 3, we considered the system at the instant of a departure and determined the lengths of the next two departure periods. To compute the transition probabilities $\mathbf{D}_{rc}$, we have to consider all cases where an interdeparture period of length r has finished and a period of length c has to be realized next, i.e. we consider the system at the departure instant in between of two interdeparture periods.

To improve the readability, we introduce the function $\phi\{a\}$ that indicates whether condition a is true or false. In addition, the implementation of the algorithm becomes straightforward. The indicator function simply has to be replaced by an if-clause.

$$\phi\{a\} \;=\; \begin{cases} 1 & \text{if} \quad a \quad \text{is true,} \\ 0 & \text{if} \quad a \quad \text{is false.} \end{cases} \tag{17}$$

The entries of matrix $\mathbf{D}$ are given by

$$\begin{aligned}
\mathbf{D}_{rc} \;=\; & \sum_{i_A=0}^{\infty} \sum_{i_B=0}^{\infty} a(i_A)\, b(i_B) \\
& \Bigg\{ \sum_{i_U=0}^{\infty} \frac{(1-p_{\text{idle}})\, b(r)\, u^v(i_U)}{d(r)} \quad \phi\Big\{ \underbrace{\big[(i_B = c) \wedge (i_A - i_U \le r)\big]}_{\text{case 1}} \vee \\
& \vee \underbrace{\big[(i_B + i_A - i_U = c + r) \wedge (i_A - i_U > r)\big]}_{\text{case 2}} \Big\} + \\
& + \sum_{i_U=-1}^{-r} \frac{p_{\text{idle}}\, b(r+i_U)\, u^v(i_U)}{d(r)} \quad \phi\Big\{ \underbrace{\big[(i_B = c) \wedge (i_A - i_U \le r)\big]}_{\text{case 3}} \vee \\
& \vee \underbrace{\big[(i_B + i_A - i_U = c + r) \wedge (i_A - i_U > r)\big]}_{\text{case 4}} \Big\} \Bigg\},
\end{aligned} \tag{18}$$

$$\text{if} \quad d(r) > 0 \quad , \quad r, c = 0, 1, 2, \ldots \ .$$

To force $\mathbf{D}$ to be stochastic, the following workaround is applied if $d(r) = 0$.

$$\mathbf{D}_{rc} = \begin{cases} 1 & \text{if} \quad c = r + 1, \\ 0 & \text{else.} \end{cases} \tag{19}$$

By means of this construction, state r becomes transient and, as expected, $d(k)$ is the left eigenvector of $\mathbf{D}$ for the eigenvalue 1.

The matrix $\mathbf{D}$ provides an alternative way to estimate the coefficient of correlation by

$$\mathrm{Corr}[D_n \, D_{n+1}] = \sum_{r=1}^{\infty} r \, d(r) \sum_{c=1}^{\infty} c \, \mathbf{D}_{rc} \, . \tag{20}$$

5 Numerical Results

To apply the above results, infinite distributions have to be approximated by finite ones. We suggest to cut off and normalize the PMF's such that the probability mass that was cut off from the original PMF is less than a given constant ϵ, say 10^{-10}. Given the PMF $x(k)$ and the accuracy constant ϵ, we obtain the following approximate PMF $\widehat{x}(k)$.

$$k_{\max} = \max_k \left\{ \sum_{i=0}^{k} x(i) < 1 - \epsilon \right\} \quad , \tag{21}$$

$$\widehat{x}(k) = \begin{cases} \frac{1}{\sum_{i=0}^{k_{\max}} x(i)} \, x(k) & \text{if} \quad k \leq k_{\max} \quad , \\ 0 & \text{else.} \end{cases} \tag{22}$$

If a continuous PDF $X(t)$ is given, it has to be approximated by a discrete PMF $\widetilde{x}(k)$ with respect to a given degree of accuracy Δt first. To this end, we suggest the following procedure.

$$\widetilde{x}(0) = X\big(\frac{\Delta t}{2}\big), \tag{23}$$

$$\widetilde{x}(k) = X\big((k + \frac{1}{2})\Delta t\big) - X\big((k - \frac{1}{2})\Delta t\big), \qquad k = 0, 1, 2, \ldots \quad . \tag{24}$$

In general, the PMF $\widetilde{x}(k)$ will have to be cut as outlined above.

Table 1: *Accuracy of the discrete-time approximation of* $\mathrm{Corr}[D_n\, D_{n+1}]$.

	closed formula	Δt		
		0.2	0.1	0.02
$M/M/1$	0	-0.00061875	-0.00015576	
$M/D/1$	0.07102044	0.07103136	0.07102333	0.07102073
$D/M/1$	-0.26536229	-0.27150670	-0.26695017	-0.26542648

Note, that in some cases it is useful to cut not only the PMF's $a(k)$ and $b(k)$ but also the PMF's $u^v(k)$ and $d(k)$ to save computation time.

In the sequel, we compare the coefficients of correlation obtained from the presented discrete-time approach to those obtained from closed formulae. There is a large body of literature on the interdeparture processes of $GI/GI/1$ queueing systems. Here, we restrict our attention to the three prominent systems $M/M/1$ [2], $M/D/1$ [4], and $D/M/1$ [3] and the lag-1 coefficients of correlation of their interdeparture times.

For the $M/M/1$ system, the output process is Poissonian with rate $1/\mathrm{E}[A]$ if the system load $\rho = \mathrm{E}[B]/\mathrm{E}[A] < 1$. As a consequence $\mathrm{Corr}[D_n\, D_{n+1}]_{MM} = 0$ in these cases.

For the $M/D/1$ system with $\rho < 1$, the interdeparture time coefficient of correlation is

$$\mathrm{Corr}[D_n\, D_{n+1}]_{MD} = \frac{e^{-\rho} + \rho - 1}{1 + \rho} . \tag{25}$$

For the $D/M/1$ system with $\rho < 1$, we obtain the output process correlation by

$$\mathrm{Corr}[D_n\, D_{n+1}]_{DM} = \frac{1}{2\delta} e^{-\frac{1}{\rho}} (\delta - 1) \quad \text{with} \quad \delta = e^{\frac{\delta - 1}{\rho}} . \tag{26}$$

To assess the accuracy of Equation (16), we compare the closed formulae results to discrete-time approximations with various discretization constants Δt. The cut-off criterion ϵ is set to 10^{-9} and the system load ρ equals 0.5.

Table 1 shows that even for coarse discretizations of the continuous PDF the resulting coefficients of correlations are very close to those obtained from closed formulae.

We checked the accuracy for each system for a variety of system loads and obtained the same quality of results.

If the value $\mathrm{Corr}[D_n\, D_{n+1}]$ is estimated from an $n \times n$ matrix $\mathbf{D}$ by Equation (20) where the same PMF's are used as for Equation (16) and $k_{max}(d) = n$ the same degree of accuracy is obtained by Equation (20) and Equation (16).

6 Conclusion

In this paper, we presented a method to compute the coefficient of correlation for two consecutive interdeparture time periods of a discrete-time $GI/GI/1$ system. In addition, we provided an algorithm to create the one-step transition probability matrix of the interdeparture time process of this system. The presented methodology can be applied to approximate the correlations of continuous-time $GI/GI/1$ systems with a high degree of accuracy.

A future study will show whether the quality of the performance measures of a tandem queueing system with arbitrary interarrival time and service time distributions can be considerably improved by making use of the correlation measures provided by our methods. In particular, we are interested in a comparison of the waiting times at the second queue if uncorrelated departures from the first queue are assumed and, on the other hand, if the interdeparture process is modeled by means of the transition probability matrix $\mathbf{D}$.

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