

# A Class of Renewal Interrupted Poisson Processes and Applications to Queueing Systems

By P. Tran-Gia<sup>1</sup>

**Abstract:** Switched Poisson Processes and Interrupted Poisson Processes are often employed to characterize traffic streams in distributed computer and communications systems, especially in investigations of overflow processes in telecommunication networks. With these processes, input streams having inter-segment correlations and high variance as well as state-dependent traffic can properly be modelled. In this paper we first derive an approximation method to describe the Generalized Switched Poisson processes in conjunction with a renewal assumption. As a special case of this class of processes, the class of Interrupted Poisson processes is also included in the investigation. As a result, a generalization of the well-known class of Interrupted Poisson processes is obtained. It is shown that the renewal property is also given for this general class of Interrupted Poisson processes having generally distributed off-phase. To illustrate the accuracy of the presented renewal approximation of Generalized Switched Poisson processes and to show the major properties of the General Interrupted Poisson processes, applications to some basic queueing systems are discussed by means of numerical results.

**Zusammenfassung:** Zur Beschreibung von Verkehrsströmen in modernen Rechner- und Kommunikationssystemen werden zunehmend Punktprozesse mit komplexen Charakteristiken benötigt. Die Klassen von unterbrochenen Poisson-Prozessen (IPP: Interrupted Poisson Process) sowie von geschalteten Poisson-Prozessen (SPP: Switched Poisson Process) finden hier häufig Anwendung, z. B. in Untersuchungen von Überlaufverkehr in Fernsprechnetzen, hochvarianzigen Datenströmen in Paketvermittlungssystemen etc. In diesem Beitrag wird zunächst der verallgemeinerte SPP-Prozess untersucht und dessen Verteilungsfunktion entwickelt, wobei eine Approximation mittels einer Erneuerungsannahme vorgenommen wird. Danach wird eine exakte Beschreibung des verallgemeinerten IPP-Prozesses vorgestellt, in der einige Prozeßbeschreibungen in der Literatur als Sonderfälle enthalten sind. Abschließend werden Parameterstudien anhand einiger einstufiger Verkehrsmodelle durchgeführt.

**Key words:** Performance analysis, Queueing models, Point process, Interrupted Poisson process, Switched Poisson process, Overflow process, Renewal approximation.

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<sup>1</sup> This work was done while the author was with Institute of Communications Switching and Data Technics, University of Stuttgart, Seidenstrasse 36, D-7000 Stuttgart 1, FRG. Current address: IBM Research Division, Zurich Research Laboratory, Säumerstrasse 4, CH-8803 Rüschlikon, Switzerland.

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## 1 Introduction

In real-time processing systems with distributed control, statistical characteristics of traffic streams resulting from events interchanged between processors are very complex. On the one hand, the behavior of these streams often depends on the actual state of the whole system, which is, e.g., represented by the number of active processes to be scheduled in the system. On the other hand, traffic streams, which result from input processes in overload situations, such as interprocess or interprocessor communications, are highly time-dependent. They have to be described by means of non-stationary processes or approximated by means of quasi-stationary processes.

For performance modeling of such systems, to describe properly traffic processes obtained by measurements, numerous renewal and non-renewal process description and approximation techniques has been proposed and investigated (cf. Fond 1978; Heffes 1976, 1980, 1986; Kuczura 1972, 1973; Machihara 1983, Meier 1984, Tran-Gia 1983). There are a number of studies using the Switched Poisson process and its related processes (e.g. the Interrupted Poisson process) as input process of queueing models. In Kuczura (1973) and Heffes (1976) systems with Interrupted Poisson input process (IPP) are discussed. While in Kuczura (1973) the infinite server model with IPP input is investigated, the output process of the queue  $GI/M/n$  is treated in Heffes (1976). The Switched Poisson process (SPP) with Markovian (exponential) phase lengths is dealt with e.g. in Fond (1978), Heffes (1980), Kuczura (1972), Meier (1984) and Yechiali (1971). In Kuczura (1972) the process SPP appears as a special case of the  $GI+M$  input. A solution for the delay system  $SPP/M/1$  is given in Yechiali (1971). To clarify the relationship between the classes of processes mentioned here, Fig. 1

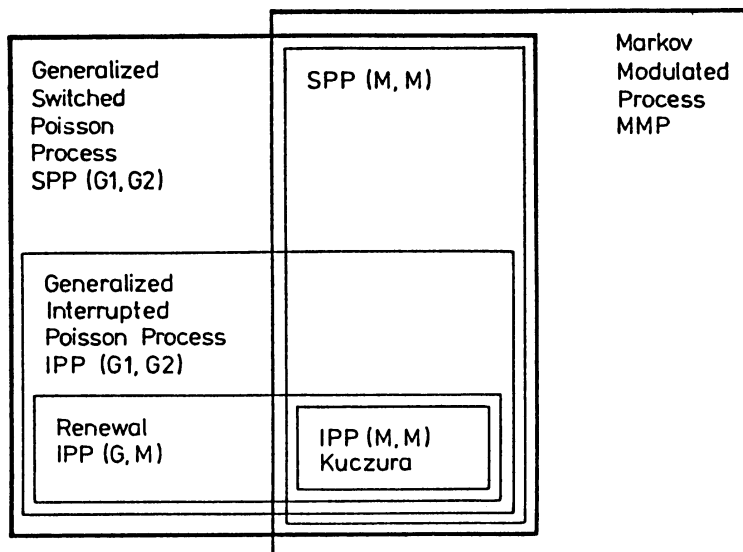


Fig. 1. Overview and relationship between classes of processes

gives an overview of these process classes, such as Switched Poisson process, Interrupted Poisson process and Markov Modulated Poisson process, etc.

Section 2 presents a generalization of the Switched Poisson Process  $SPP(G_1, G_2)$ , for which a general Laplace-Stieltjes-Transform formula based on a renewal approximation has been derived (cf. Tran-Gia 1983). The formula obtained in Section 2 reconciles a wide range of results known in the literature. In Section 3, attention is devoted to a particular case of  $SPP(G_1, G_2)$ , the class of Interrupted Poisson processes  $IPP(G, M)$ . It is shown that this class of processes, which is a generalization of the Interrupted Poisson process given by Kuczura (1973), forms a class of general traffic processes with the renewal property. Finally, Section 4 investigates the presented classes of processes in conjunction with standard queueing systems, in order to estimate the applications to more complex modeling approaches.

## 2 The Generalized Switched Poisson Process

In this section, the Generalized Switched Poisson Process will be defined and a characterization of the process in terms of interarrival distribution functions using a renewal approximation will be presented.

### 2.1 Process Description

A generating model of the Generalized Switched Poisson process is shown in Fig. 2. The process results from an alternated switching between two Poisson processes (the originating processes), which are characterized by the rates  $\lambda_1$  and  $\lambda_2$  respectively. The visit times of the resulting process are independent and identically distributed random variables  $T_1$  and  $T_2$ .

According to the arbitrary phase length distribution functions, the following notation will be used for the class of Generalized Switched Poisson processes:  $SPP(G_1, G_2)$ ,

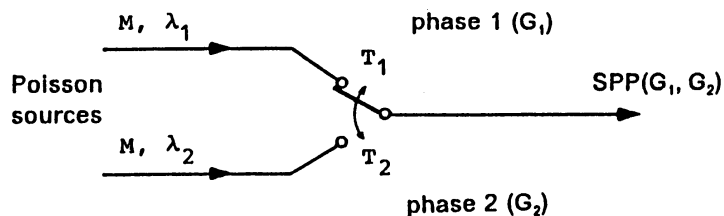


Fig. 2. Generating model for the Generalized Switched Poisson Process

where  $G_1$  and  $G_2$  denote the distribution types of  $T_1$  (Phase 1) and  $T_2$  (Phase 2), respectively. For  $\lambda_1 = 0$ , we have the special case of the class of General Interrupted Poisson processes IPP( $G_1, G_2$ ).

Thus, the Generalized Switched Poisson Process SPP( $G_1, G_2$ ) can be completely characterized by the following random variables (r.v.):

- $T_1$  r.v. for the length of phase 1 with mean  $1/\omega_1$
- $T_2$  r.v. for the length of phase 2 with mean  $1/\omega_2$
- $T_{A1}$  r.v. for the interarrival time during phase 1 with mean  $1/\lambda_1$
- $T_{A2}$  r.v. for the interarrival time during phase 2 with mean  $1/\lambda_2$

According to the definition,  $T_{A1}$  and  $T_{A2}$  are negative exponentially distributed r.v. corresponding to the originating Poisson sources in Fig. 2, where

$$F_{Ai}(t) = \Pr\{T_{Ai} \leq t\} = 1 - e^{-\lambda_i t}; \quad E[T_{Ai}] = \frac{1}{\lambda_i}, \quad i = 1, 2. \quad (1)$$

Furthermore, the mean phase lengths are denoted by:

$$E[T_i] = h_i = \frac{1}{\omega_i}, \quad i = 1, 2. \quad (2)$$

As an alternative to the basic parameters given in eqs. (1, 2) the following process parameters are defined for modeling purposes which allow a description of input processes (e.g., overload traffic streams) in a more realistic way:

i) The mean arrival rate

$$\lambda = \frac{\lambda_1 h_1 + \lambda_2 h_2}{h_1 + h_2} = \frac{\lambda_1 \omega_2 + \lambda_2 \omega_1}{\omega_1 + \omega_2}. \quad (3)$$

ii) Considering two consecutive phases of type 1 and 2 together as a period of the process, the mean number of events in a period is

$$n_0 = \lambda_1 h_1 + \lambda_2 h_2 = \frac{\lambda_1}{\omega_1} + \frac{\lambda_2}{\omega_2} \quad (4)$$

The parameters  $\lambda$  and  $n_0$  can be used to characterize the switching frequency of the SPP( $G_1, G_2$ ) process.

iii) The ratio of phase lengths

$$\Theta = \frac{h_2}{h_1} = \frac{\omega_1}{\omega_2}. \tag{5}$$

iv) The overload factor

$$\gamma = \frac{\lambda_2}{\lambda}. \tag{6}$$

In the following it is assumed that  $\gamma \geq 1$  ( $\lambda_2 \geq \lambda$ ), i.e.  $\lambda_2$  represents the higher and  $\lambda_1$  the lower load level.

In this context, two well-known renewal processes, the Interrupted Poisson Process IPP( $M, M$ ) (cf. Kuczura 1973) and the Poisson process can be identified as limiting cases of the Switched Poisson Process. These boundary processes correspond to the limiting values of the overload factor  $\gamma$ :

- Poisson process, which corresponds to the minimum value  $\gamma_{\min}$  of the overload factor:

$$\lambda_1 = \lambda_2 = \lambda \rightarrow \gamma_{\min} = 1. \tag{7}$$

- Interrupted Poisson process IPP( $M, M$ ), which corresponds to the maximum value  $\gamma_{\max}$  of the overload factor:

$$\lambda_1 = 0 \rightarrow \gamma_{\max} = \frac{\omega_1 + \omega_2}{\lambda_2 \omega_1} = 1 + \frac{1}{\Theta}. \tag{8}$$

## 2.2 Renewal Approximation for the Generalized Switched Poisson Process

In general the class of Generalized Switched Poisson Process  $SPP(G_1, G_2)$  is non-renewal. Under assumption of the renewal property, an approximate expression for the Laplace-Stieltjes-transform is developed in Tran-Gia (1983). The derivation will briefly be described, ensuing the following steps:

- i) Calculation of the distribution function of the forward recurrence interarrival time
- ii) Calculation of the interarrival distribution function using the renewal assumption.

### 2.2.1 Notation

For the notation of random variables and their related functions, the following symbols will be used:

$T_i$  random variable (r.v.), index  $i$

$T_i^f$  forward recurrence time of the r.v.  $T_i$

$F_i(t) = \Pr\{T_i \leq t\}$  probability distribution function (PDF) of the r.v.  $T_i$

$f_i(t) = \frac{dF_i(t)}{dt}$  probability density function (pdf) of the r.v.  $T_i$

$\Phi_i(s) = \text{LT}\{f_i(t)\} = \text{LST}\{F_i(t)\}$  Laplace-Stieltjes-transform of the PDF  $F_i(t)$  or Laplace-Transform of the pdf  $f_i(t)$

Additionally, the conditional random variables  $T_i|T_i > T_j$  and  $T_i|T_j > T_i$  are introduced, which have the following *unnormalized* pdf's and Laplace-transforms:

$$\begin{aligned} \tilde{f}_i(t)|_{T_i > T_j} &= f_i(t)F_j(t), & \tilde{\Phi}_i(s)|_{T_i > T_j} &= \text{LT}\{\tilde{f}_i(t)|_{T_i > T_j}\} \\ \tilde{f}_i(t)|_{T_j > T_i} &= f_i(t)(1 - F_j(t)), & \tilde{\Phi}_i(s)|_{T_j > T_i} &= \text{LT}\{\tilde{f}_i(t)|_{T_j > T_i}\} \end{aligned} \quad (9)$$

The corresponding normalized forms are:

$$f_i(t)|_{T_i > T_j} = \frac{\tilde{f}_i(t)|_{T_i > T_j}}{\int_0^\infty \tilde{f}_i(t)|_{T_i > T_j} dt}, \quad \Phi_i(s)|_{T_i > T_j} = \text{LT}\{f_i(t)|_{T_i > T_j}\} \tag{10}$$

$$f_i(t)|_{T_j > T_i} = \frac{\tilde{f}_i(t)|_{T_j > T_i}}{\int_0^\infty \tilde{f}_i(t)|_{T_j > T_i} dt}, \quad \Phi_i(s)|_{T_j > T_i} = \text{LT}\{f_i(t)|_{T_j > T_i}\}$$

2.2.2 Forward Recurrence Time Distribution Function

The calculation of the PDF  $F^r(t)$  of the forward recurrence time and its Laplace-Stieltjes-transform  $\Phi^r(s)$  is based on an observation of the process at an arbitrary time epoch  $t^*$  (the observation point, see Fig. 3). The probability of seeing the process in a phase 1 can be written as follows:

$$p_1 = \Pr\{t^* \text{ is in a phase of type 1}\} = \frac{h_1}{h_1 + h_2} = \frac{\omega_2}{\omega_1 + \omega_2} \tag{11}$$

and analogously:

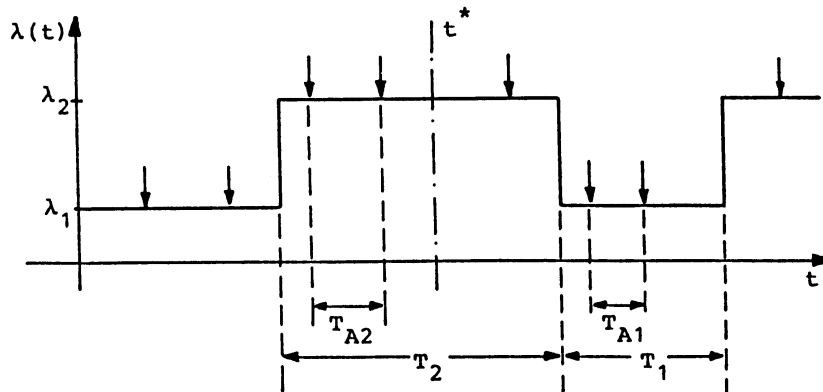


Fig. 3. Parameters of the Generalized Switched Poisson Process

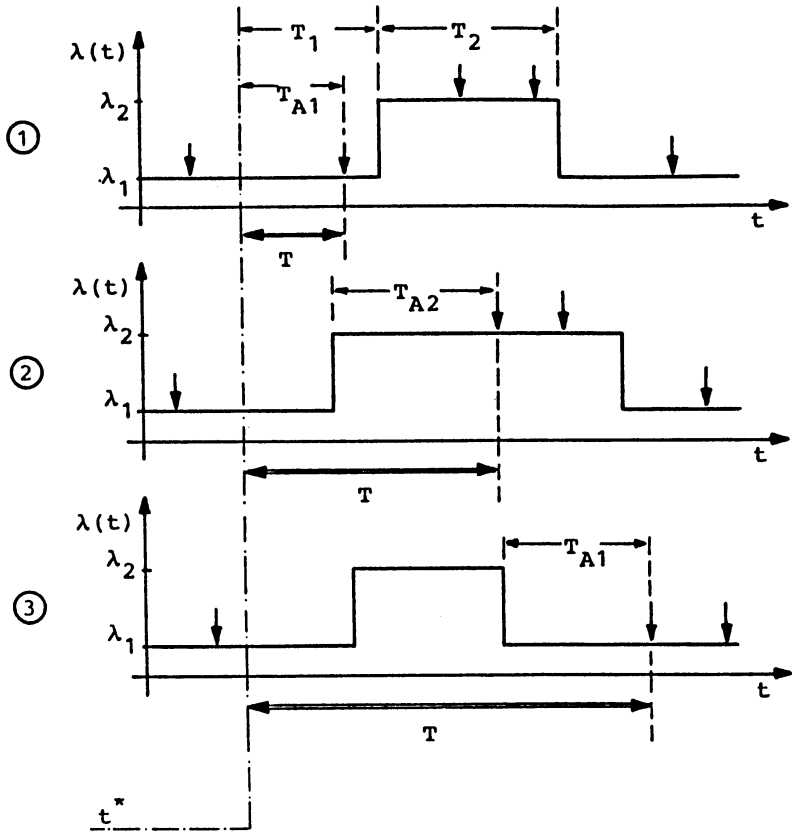


Fig. 4. Calculation of the forward recurrence time of the Generalized Switched Poisson Process

$$p_2 = \Pr\{t^* \text{ is in a phase of type 2}\} = \frac{h_2}{h_1 + h_2} = \frac{\omega_1}{\omega_1 + \omega_2} \quad (12)$$

Let the observation point  $t^*$  be now in a phase 1. Taking into account this assumption, Fig. 4 illustrates three examples of the forward recurrence time  $T^r$ , which is the duration from  $t^*$  to the next arrival instant.

Two cases can occur:

- i) The expected event is an arrival in the current phase 1 (see case 1 in Fig. 4). In this case the forward recurrence time is

$$T^r = T_{A1} | T_1^r > T_{A1}$$



- ii) The expected event is the end of the current phase 1. The process must spend the time interval  $T_1^r | T_{A1} > T_1$ , after which the process observation will be continued until the next arrival is found.

The process is now observed from the beginning of a phase 2. The following cases can occur:

- i) The expected event is an arrival in the current phase 2 (see case 2 in Fig. 4). In this case, the compound forward recurrence time is given as

$$T^r = T_1^r | T_{A1} > T_1 + T_{A2} | T_2 > T_{A2}$$

- ii) The expected event is the end of the current phase 2, i.e. no arrival has occurred during this phase. After the phase  $T_2 | T_{A2} > T_2$  the observation of the process will be continued.

The process is being observed at the beginning of a phase 1, where the following two cases can occur:

- i) The expected event is an arrival in the current phase 1 (see case 3 in Fig. 4). The compound forward recurrence time is given according to

$$T^r = T_1^r | T_{A1} > T_1 + T_2 | T_{A2} > T_2 + T_{A1} | T_1 > T_{A1}$$

- ii) The end of the current phase 1 is reached and no arrival has been registered.

The observation of the process can be analogously continued until an arrival is attained. Taking into account all combinatorial possibilities for the forward recurrence time  $T^r$ , a phase diagram as shown in Fig. 5 can be obtained. It should be recalled here that the observation point  $t^*$  is assumed to be in a phase of type 1.

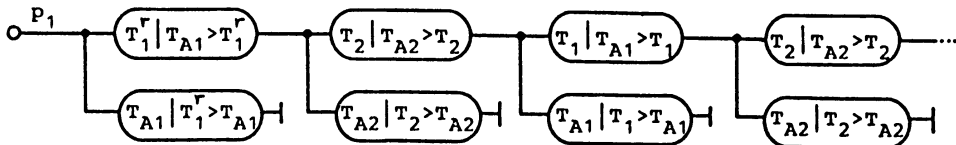


Fig. 5. Phase diagram of the forward recurrence time (conditioned on an observation point in phase 1)

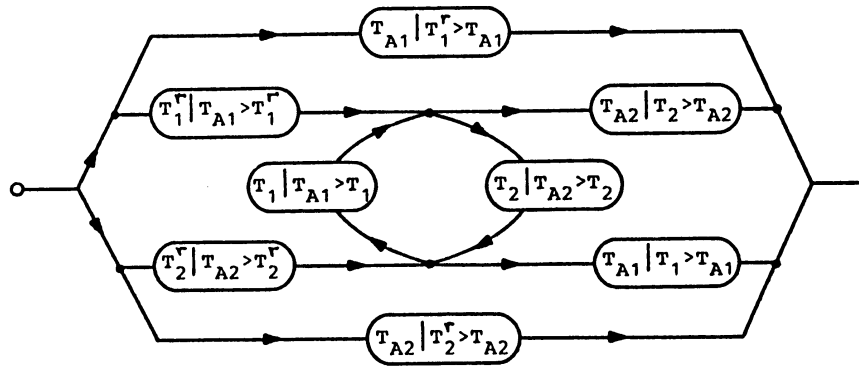


Fig. 6. Phase diagram of the forward recurrence time

The combination of the two cases for the observation point  $t^*$  yields to the complete phase diagram of the forward recurrence interarrival time in Fig. 6, where the random variables for time intervals are indicated.

Considering the phase diagram in Fig. 6 as a Mason flow graph (cf. Mason 1953, 1956), which consists of six forward paths and one loop, the Laplace-Stieltjes-transform of the forward recurrence interarrival PDF of the Generalized Switched Poisson Process  $SPP(G_1, G_2)$  is obtained as follows:

$$\begin{aligned}
 \Phi^r(s) &= p_1 \tilde{\Phi}_{A1}(s) | T_1^r > T_{A1} \\
 &+ p_1 \tilde{\Phi}_1^r(s) | T_{A1} > T_1^r \frac{\tilde{\Phi}_{A2}(s) | T_2 > T_{A2} + \tilde{\Phi}_2(s) | T_{A2} > T_2 \tilde{\Phi}_{A1}(s) | T_1 > T_{A1}}{1 - \tilde{\Phi}_1(s) | T_{A1} > T_1 \tilde{\Phi}_2(s) | T_{A2} > T_2} \\
 &+ p_2 \tilde{\Phi}_{A2}(s) | T_2^r > T_{A2} \\
 &+ p_2 \tilde{\Phi}_2^r(s) | T_{A2} > T_2^r \frac{\tilde{\Phi}_{A1}(s) | T_1 > T_{A1} + \tilde{\Phi}_1(s) | T_{A1} > T_1 \tilde{\Phi}_{A2}(s) | T_2 > T_{A2}}{1 - \tilde{\Phi}_1(s) | T_{A1} > T_1 \tilde{\Phi}_2(s) | T_{A2} > T_2}
 \end{aligned} \tag{13}$$

The probabilities  $p_1, p_2$  are given in eqs. (11), (12) and the Laplace-Stieltjes-transforms of the conditional PDFs for the conditional phases in eq. (13) can be calculated according to eqs. (9) and (10). It should be recalled here that the expression for  $\Phi^r(s)$  given in eq. (13) is valid for arbitrary types of the phase lengths  $T_1$  and  $T_2$ .

### 2.2.3 Renewal Approximation

Assuming the renewal property for the Generalized Switched Poisson Process  $SPP(G_1, G_2)$ , we obtain the following approximate expression for the Laplace-Stieltjes-transform of the interarrival distribution function (see Cox 1962):

$$\Phi(s) = 1 - \frac{s}{\lambda} \Phi^r(s) \tag{14}$$

where  $\Phi^r(s)$  is given in eq. (13).

### 2.3 Special Case of Exponential Phase Lengths

As mentioned above, the expressions given in eqs. (13) and (14) can be used for arbitrary phase lengths  $T_1$  and  $T_2$  (cf. Figs. 2 and 3), which correspond to the notation  $SPP(G_1, G_2)$ . In the following we will devote attention to a special case, which is often used in the literature. The phase lengths  $T_i$  ( $i = 1, 2$ ) are now negative exponentially distributed:

$$F_i(t) = \Pr\{T_i \leq t\} = 1 - e^{-\omega_i t}, \quad i = 1, 2. \tag{15}$$

Using the notation presented before, the process is of type  $SPP(M, M)$ . This special case of the Generalized Switched Poisson Process corresponds to the two-state Markov Modulated Poisson Process (MMP) discussed in Heffes (1980, 1986), Meier (1984) (cf. Fig. 1) and the input process with heterogeneous arrivals analyzed in Yechiali (1971).

According to the PDFs given in eqs. (1) and (15) for the r.v.  $T_1, T_2, T_{A1}, T_{A2}$  of the process  $SPP(M, M)$ , the Laplace-Stieltjes-transforms of the conditional PDFs in eq. (9) are determined and subsequently, the Laplace-Stieltjes-transform of the forward recurrence time:

$$\Phi^r(s) = \frac{1}{\omega_1 + \omega_2} \frac{\lambda_1 \omega_2 (s + \omega_1 + \omega_2 + \lambda_2) + \lambda_2 \omega_1 (s + \omega_1 + \omega_2 + \lambda_1)}{(s + \lambda_1)(s + \lambda_2) + \omega_1 (s + \lambda_2) + \omega_2 (s + \lambda_1)}. \tag{16}$$

Taking into account the renewal assumption in eq. (14), the Laplace-Stieltjes-transform of the interarrival time of the Generalized Switched Poisson Process with exponential phase lengths  $SPP(M, M)$  is given by:

$$\Phi(s) = \frac{1}{\lambda_1 \omega_2 + \lambda_2 \omega_1} \frac{s(\lambda_1^2 \omega_2 + \lambda_2^2 \omega_1) + (\lambda_1 \lambda_2 + \lambda_1 \omega_2 + \lambda_2 \omega_1)(\lambda_1 \omega_2 + \lambda_2 \omega_1)}{s^2 + s(\lambda_1 + \lambda_2 + \omega_1 + \omega_2) + \lambda_1 \lambda_2 + \lambda_1 \omega_2 + \lambda_2 \omega_1}. \quad (17)$$

The corresponding pdf of the interarrival time can be obtained from eq. (17)

$$f(t) = \text{LT}^{-1}\{\Phi(s)\} = \frac{K}{s_1 + s_2} [(a - s_1)e^{-s_1 t} + (s_2 - a)e^{-s_2 t}] \quad (18)$$

where

$$K = \frac{\lambda_1^2 \omega_2 + \lambda_2^2 \omega_1}{\lambda_1 \omega_2 + \lambda_2 \omega_1}, \quad a = \frac{\lambda_1 \lambda_2 + \lambda_1 \omega_2 + \lambda_2 \omega_1}{K}$$

$$s_{1,2} = \frac{b}{2} \pm \frac{1}{2} \sqrt{b^2 - 4aK} \quad \text{with} \quad b = \lambda_1 + \lambda_2 + \omega_1 + \omega_2.$$

While the mean interarrival time is

$$E[T] = \frac{1}{\lambda} \quad (19a)$$

as expected, the coefficient of variation  $c$  can be determined from eq. (17):

$$c^2 = 2 \frac{\lambda(\lambda_1 + \lambda_2 + \omega_1 + \omega_2 - \lambda)}{\lambda_1 \lambda_2 + \lambda_1 \omega_2 + \lambda_2 \omega_1} - 1. \quad (19b)$$

As mentioned above, the formula given in eq. (17) covers the whole range between the Poisson process and the Interrupted Poisson process, according to the overload factor  $\gamma$  defined in eq. (6). For these two boundary processes, which have the renewal property, eq. (17) can be rewritten as follows:

- Poisson process:  $\lambda_1 = \lambda_2 = \lambda$  ( $\gamma = \gamma_{\min} = 1$ );  $\Phi(s) = \frac{\lambda}{s + \lambda}$
- Interrupted Poisson process IPP( $M, M$ ):  $\lambda_1 = 0$  ( $\gamma = \gamma_{\max} = 1 + \frac{1}{\Theta}$ ):

$$\Phi(s) = \frac{\lambda_2(s + \omega_1)}{s^2 + s(\omega_1 + \omega_2 + \lambda_2) + \lambda_2 \omega_1} \quad (\text{cf. Kuczura 1973}) \quad (20)$$

### 2.4 Accuracy of the Renewal Approximation

In order to estimate the accuracy of the renewal approximation, we consider in this chapter the process SPP( $M, M$ ) as input of a single exponential server queueing system with finite waiting capacity  $S$ , i.e. the delay-loss system SPP( $M, M$ )/ $M$ /1- $S$ . The mean service time will be used here to standardize the results.

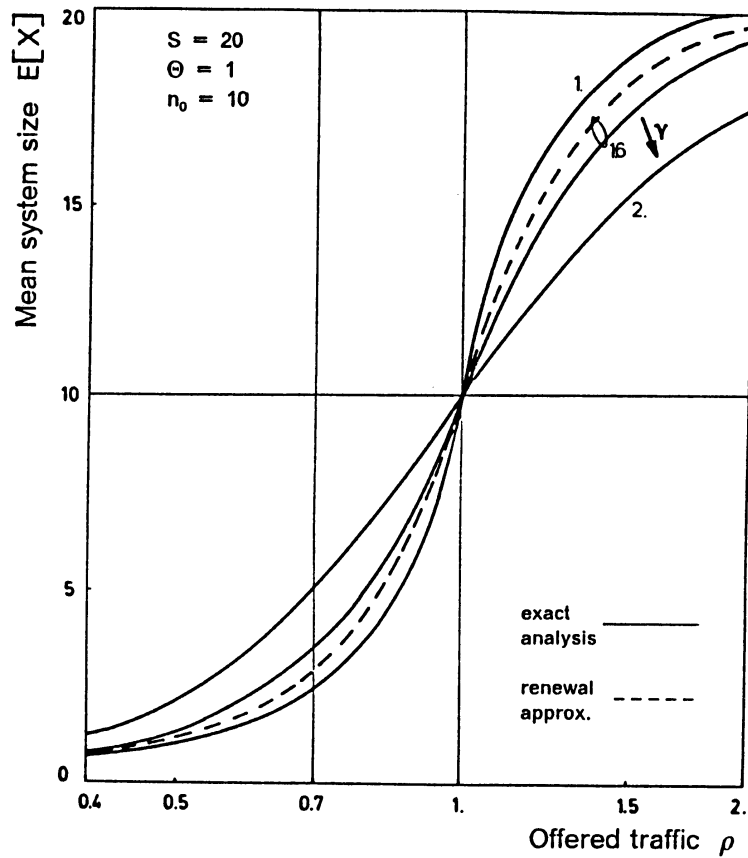


Fig. 7. Accuracy of the renewal approximation: mean system size vs offered traffic

In the following, system characteristics will be compared to validate the renewal approach and to show the dependency of the approximation accuracy on the process parameters, where:

- i) The exact solution of the system  $SPP(M, M)/M/1-S$  is carried out by means of a two-dimensional Markov process. The results are obtained using a recursive algorithm.
- ii) The renewal approximation considers the process  $SPP(M, M)$  as a general independent input process with the Laplace transform given in eq. (17) or the PDF in eq. (18). The results for the approximation are obtained by solving the arising system  $GI/M/1-S$ , using a numerical algorithm.

Fig. 7 shows the mean system size as a function of the offered traffic  $\rho = \frac{\lambda}{\mu}$ . For the chosen parameters ( $S = 20, \Theta = 1, n_0 = 10$ ) the overload factor  $\gamma$  can vary from  $\gamma_{min}$

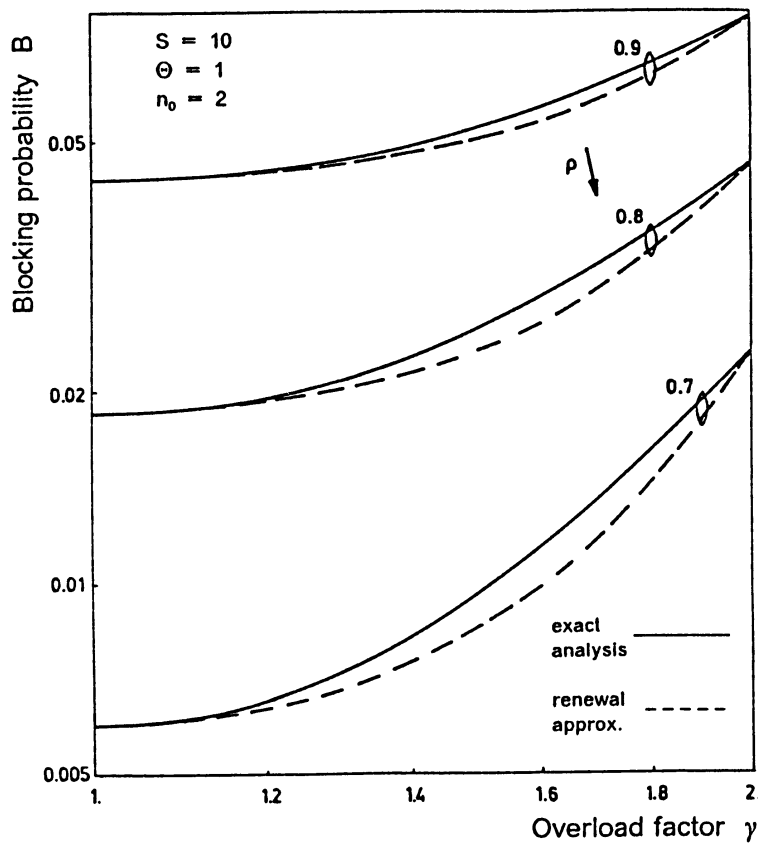


Fig. 8. Accuracy of the renewal approximation. blocking probability vs overload factor

= 1 (Poisson process) to  $\gamma_{\max} = 1 + \frac{1}{\Theta} = 2$  (Interrupted Poisson process). For these two boundary cases, the renewal assumption is exact as expected.

The blocking probabilities are depicted in Fig. 8. For different values of the offered traffic, it is seen here that the renewal assumption is a closed approximation for a wide range of  $\gamma$ . However, the accuracy shown here depends very strongly on the mean number  $n_0$  of the arrivals per process period. This can be explained by the fact that for smaller values of  $n$ , the Switched Poisson process is of more random nature and therefore the renewal approximation is more accurate.

### 3 General Interrupted Poisson Processes with Renewal Property

#### 3.1 Process Description

In the case of  $\lambda_1 = 0$  we obtain from the  $SPP(G_1, G_2)$  a generic form of the Interrupted Poisson Process, denoted in the following by  $IPP(G_1, G_2)$ . Choosing further  $T_1$  and  $T_2$  to be negative exponentially distributed, we arrive at the well-known special case of this class of traffic processes, the  $IPP(M, M)$  process (cf. Heffes 1980, 1986; Meier 1984), which are often employed in the teletraffic theory to model bursty traffic streams like overflow processes.

Assuming the on-phase of the process to be exponential, while the off-phase remains general, we obtain a new class of Interrupted Poisson processes which is of renewal nature. We denote this class as General Interrupted Poisson Process, characterized by  $IPP(G, M)$ . The renewal property of this process class can be shown using relationships of doubly stochastic Poisson processes as given by Kingman (1964).

#### 3.2 Interarrival Distribution Function

Using results derived in eqs. (13) and (14) the interarrival distribution function of the General Interrupted Poisson Process  $IPP(G, M)$  can be obtained as follows:

$$\Phi_{IPP(G, M)}(s) = \frac{\lambda_2}{s + \lambda_2 + \omega_2(1 - \Phi_1(s))} \tag{21}$$

where  $\Phi_1(s)$  is the Laplace-Stieltjes-transform of the off-phase of the process.

### 3.3 Special Cases

Various results on Interrupted Poisson Processes known in the literature can be considered as special cases of the result in eq. (21), e.g.

- The  $IPP(M, M)$  process which has been investigated by Kuczura (1973)
- The  $IPP(H_2, M)$  process which is presented in Machihara (1983). Opposite to the approximate result in Machihara (1983), the result we obtain in this paper following eq. (21) is exact.

### 3.4 The $IPP(G, M)/M/n$ Queue

To show some properties of the General Interrupted Poisson Process  $IPP(G, M)$ , in particular to illustrate the impact of the off-phase characteristics in conjunction with a queueing system, we take the  $IPP(G, M)$  process as input of some standard queueing systems. The delay system  $IPP(G, M)/M/n$  and the delay-loss system  $IPP(G, M)/M/n-S$  are considered in the following.

#### 3.4.1 Infinite Capacity System $IPP(G, M)/M/n$

Due to the interarrival distribution function given in Laplace-Stieltjes domain, the analysis of this system is done by means of the method given by Takacs (1962).

In Fig. 9 the impact of the off-phase interval on the sojourn time of a  $IPP(G, M)/M/n$  with infinite capacity of waiting places is shown, for different values of the number  $n$  of servers. Standard types are taken for the distribution function of the off-phase: deterministic ( $D$ ), Erlangian of 2nd order ( $E_2$ ), exponential ( $M$ ), and hyper-exponential of 2nd order ( $H_2, c = 2$ ). It can be clearly seen here that the waiting time and subsequently, the sojourn time in the system, are strongly affected by the properties of the off-phase of the General Interrupted Poisson process offered.



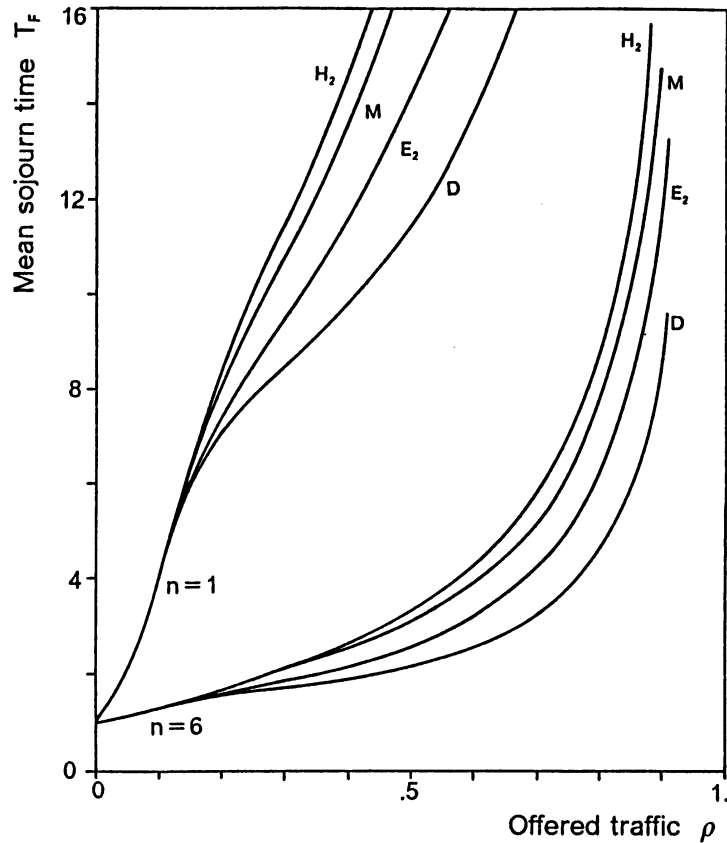


Fig. 9. Infinite capacity system  $IPP(G, M)/M/n$ : influence of the off-phase distribution of the input process on the mean sojourn time

### 3.4.2 Finite Capacity System $IPP(G, M)/M/n-S$

Since the system is now of finite capacity and the arrival process is given as of Laplace-Stieltjes transform, an imbedded Markov chain is used in the analysis. To obtain the transition probabilities, we make use of a Laplace transform inversion technique.

The blocking probability of the  $IPP(G, M)/M/3-S$  system is depicted in Fig. 10 as function of the normalized offered traffic intensity. For the off-phase the same types of distribution functions as in the case of infinite capacity systems are used. The influence of the off-phase interval on the system behavior can be again observed in Fig. 10, for different values of the number  $S$  of waiting places.

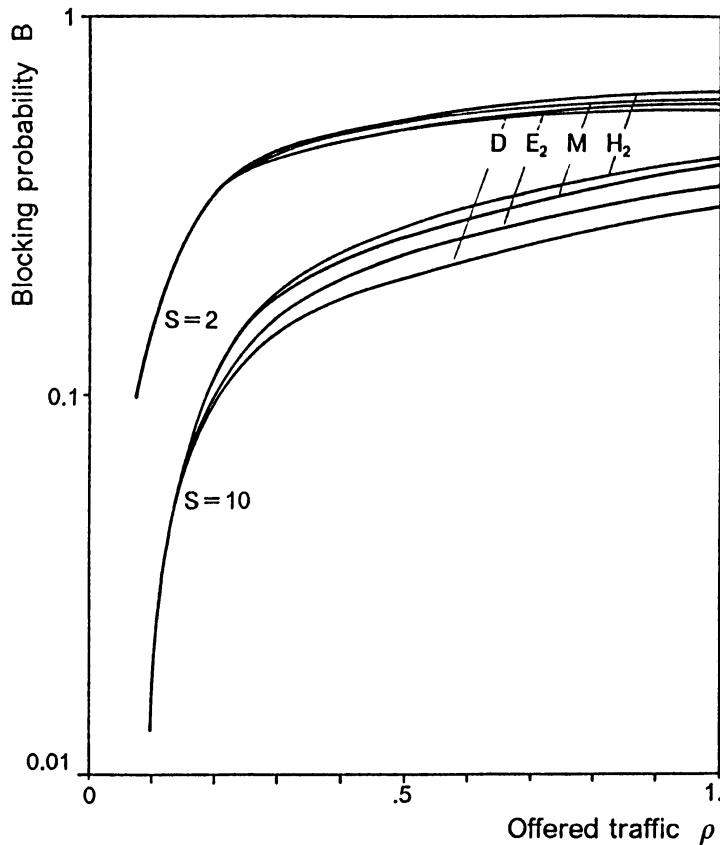


Fig. 10. Finite capacity system  $IPP(G, M)/M/n-S$ : influence of the off-phase distribution of the input process on the blocking probability

#### 4 Conclusions

In this paper we presented an approximation method to describe the Generalized Switched Poisson processes  $SPP(G_1, G_2)$ , using a renewal assumption. As shown by the results presented for the example system  $SPP(M, M)/M/1-S$  the renewal approximate technique provided is accurate for a wide range of process parameters. The renewal approximation approach represents a simplification of the analysis in the case of more complex models, where merely the distribution function of the input process or its Laplace-Stieltjes-transform is required.

We derived further a generalization of the Interrupted Poisson processes as investigated e.g. by Kuczura (1973). It is shown that this new class of General Interrupted Poisson Process  $IPP(G, M)$  is of renewal nature.

The traffic processes investigated in this paper are often employed to characterize traffic streams in distributed computer and communications systems, especially in investigations of overflow processes in telecommunication networks. With these processes, effects like inter-segment correlation of input streams and state-dependency of traffic flows can properly be modelled.

The approximation accuracy of the renewal assumption applied to the Generalized Switched Poisson process some properties of the class of General Interrupted Poisson processes are discussed by means of numerical results in conjunction with a number of basic queueing systems.

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